Mixed norm Fourier restriction estimates for surfaces in $\mathbb{R}^3$ and applications to PDEs

Dissertation

zur Erlangung des Doktorgrades
der Mathematisch-Naturwissenschaftlichen Fakultät
der Christian-Albrechts-Universität
zu Kiel

vorgelegt von

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Kiel,
Juli 2020

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Tag der mündlichen Prüfung:	22.10.2020
Zum Druck genehmigt:	22.10.2020
Dekan:	Prof. Dr. Frank Kempken

Posvećeno baki Ankici (†2019) i Dariju Mičiću (†2020)

Summary

The Fourier restriction problem for a submanifold S in $\mathbb{R}^n$ asks for which exponents $p, q \in [1, \infty]$ the Fourier restriction operator R, defined by

$$Rf := \widehat{f}\,|_S,$$

is bounded from $L^p(\mathbb{R}^n, \mathrm{d}x)$ to $L^q(S, \rho\mathrm{d}S)$. Here $\widehat{f}$ denotes the Fourier transform of f, $\mathrm{d}S$ the Riemannian surface measure of S, and ρ a fixed $C_c^\infty(S)$ function. Aside from being one of the central problems in harmonic analysis the Fourier restriction problem plays an important role in a variety of other areas of mathematics such as the theory of nonlinear dispersive equations, geometric measure theory, and number theory. It was originally introduced by E. M. Stein around 1970 and since then a lot of deep work has been done on this problem by many renowned mathematicians, including Fields Medalists C. Fefferman, J. Bourgain, and T. Tao.

Though the restriction problem for curves with nonvanishing curvature in $\mathbb{R}^2$ was already solved in the early 1970s through contributions by C. Fefferman, E. M. Stein, and A. Zygmund, it remains wide open even for the sphere in $\mathbb{R}^3$. The first result for higher dimensions, obtained by P. A. Tomas and E. M. Stein in the 1970s, was a sharp $L^p - L^2$ restriction estimate for the unit sphere in $\mathbb{R}^n$. These $L^p - L^2$ restriction estimates (also called Stein-Tomas type estimates) are much easier to handle than the general $L^p - L^q$ estimates. Indeed, by the R^*R method they reduce to $L^p - L^{p'}$ estimates for the convolution operator with integral kernel $\widehat{\rho\mathrm{d}S}$. If we can locally represent S as a graph of a function ϕ, then the integral kernel can be written as an oscillatory integral

$$\xi \mapsto \int_{\mathbb{R}^{n-1}} e^{-i(x'\cdot\xi' + \phi(x')\xi_n)} a(x')\mathrm{d}x', \qquad \xi \in \mathbb{R}^n,$$

where $a \in C_c^\infty(\mathbb{R}^{n-1})$ and we split $\xi = (\xi', \xi_n)$, $\xi' \in \mathbb{R}^{n-1}$, $\xi_n \in \mathbb{R}$. The decay rate for oscillatory integrals as above was studied by the school of V. I. Arnold, which highlighted the importance of Newton polyhedra. Of particular importance for this thesis is an algorithm developed by A. N. Varchenko in the 1970s yielding a way to calculate the decay rate when ϕ is a two-dimensional real analytic function.

In this thesis we shall be concerned with obtaining mixed norm $L^p - L^2$ Fourier restriction estimates for surfaces in $\mathbb{R}^3$. More precisely, we want to determine for which $(p_1, p_3) \in [1, 2]^2$ the operator R can be extended to an $L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)}) \to L^2(\rho\mathrm{d}S)$ bounded operator. Such mixed norm Lebesgue spaces are of great interest in PDE theory.

In the first part of this thesis we prove an extension of a more recent result of I. A. Ikromov and D. Müller to the mixed norm case. Ikromov and Müller considered local

$L^p - L^2$ Fourier restriction estimates, i.e., estimates where one takes ρ to be supported in a sufficiently small neighbourhood of a given point, and obtained sharp results for a wide range of surfaces in $\mathbb{R}^3$, including all analytic ones. We shall restrict ourselves though to surfaces given as graphs of functions ϕ which are either in adapted coordinates, or in non-adapted coordinates and have the so called linear height strictly less than two. In our proof we build on the methods of Ikromov and Müller and combine them with a method from a paper of J. Ginibre and G. Velo useful for handling mixed norm estimates.

In the second part of this thesis we prove a wide generalization of a recent result of A. Carbery, C. E. Kenig, and S. N. Ziesler, who considered the Fourier restriction problem with a mitigating factor for surfaces in $\mathbb{R}^3$ given as graphs of homogeneous two-dimensional polynomials. The estimates they considered were global, i.e., the function ρ took the role of a weight function whose support extends to the whole surface S. Compared to the proof of Carbery, Kenig, and Ziesler, our approach is considerably more elementary, avoiding any use of algebraic topology and algebraic geometry, and it allows one to cover a considerably larger class of surfaces, namely, the class of all surfaces which can be given as graphs of mixed homogeneous functions. An important step in our work is a classification of singularities for such surfaces, after which we can apply the previously developed elaborate methods and ideas of Ikromov and Müller and of Ginibre and Velo.

Finally, as an application of the above results we use the Christ-Kiselev lemma in order to obtain new Strichartz estimates for large classes of partial differential and pseudodifferential operators.

Zusammenfassung

Das Fourierrestriktionsproblem für eine Untermannigfaltigkeit S im $\mathbb{R}^n$ ist die Frage, für welche Exponenten $p, q \in [1, \infty]$ der Fourierrestriktionsoperator R, der durch

$$Rf := \widehat{f}\,|_S$$

definiert ist, einen beschränkten Operator von $L^p(\mathbb{R}^n, \mathrm{d}x)$ nach $L^q(S, \rho \mathrm{d}S)$ darstellt. Hier bezeichnet $\widehat{f}$ die Fouriertransformierte von f, $\mathrm{d}S$ das Riemannsche Volumenmaß von S und ρ eine $C_c^\infty(S)$-Funktion. Das Fourierrestriktionsproblem stellt nicht nur eines der zentralen Probleme der harmonischen Analysis dar, sondern es spielt auch eine wesentliche Rolle in vielen anderen Bereichen der Mathematik, wie z. B. in der Theorie der dispersiven Differentialgleichungen, in der geometrischen Maßtheorie und in der Zahlentheorie. Das Problem wurde ursprünglich um das Jahr 1970 von Stein eingeführt. Seitdem fanden durch die Arbeiten von herausragenden Mathematikern, u. a. von Fields-Medaillen-Trägern C. Fefferman, J. Bourgain und T. Tao, zahlreiche tiefgreifende Entwicklungen statt.

Obwohl das Restriktionsproblem für Kurven im $\mathbb{R}^2$ mit nicht verschwindender Gaußscher Krümmung schon in den frühen 1970ern durch die Arbeiten von C. Fefferman, E. M. Stein und A. Zygmund gelöst wurde, ist es in höheren Dimensionen, sogar für die Sphäre im $\mathbb{R}^3$, weit offen. Das erste Resultat für höhere Dimensionen, das P. A. Tomas and E. M. Stein in den 1970ern erhalten haben, war eine scharfe $L^p - L^2$-Restriktionsabschätzung für die Einheitssphäre im $\mathbb{R}^n$. Diese $L^p - L^2$-Restriktionsabschätzungen (auch Abschätzungen vom Stein-Tomas-Typ genannt) sind deutlich einfacher als die allgemeinen $L^p - L^q$-Abschätzungen zu behandeln. Und zwar werden sie durch die R^*R Methode auf $L^p - L^{p'}$-Abschätzungen für Faltungsoperatoren mit Integralkern $\widehat{\rho \mathrm{d}S}$ reduziert. Können wir S lokal als den Graphen einer Funktion ϕ darstellen, so nimmt der Integralkern die Form des oszillierenden Integrals

$$\xi \mapsto \int_{\mathbb{R}^{n-1}} e^{-i(x' \cdot \xi' + \phi(x') \xi_n)} a(x') \mathrm{d}x', \qquad \xi \in \mathbb{R}^n,$$

an, wobei $a \in C_c^\infty(\mathbb{R}^{n-1})$ und wir $\xi = (\xi', \xi_n)$, $\xi' \in \mathbb{R}^{n-1}$, $\xi_n \in \mathbb{R}$ schreiben. Die Abklingrate von solchen oszillierenden Integralen wurde von der Schule von V. I. Arnold studiert, die die Bedeutung von Newtonschen Polyedern hervorgehoben hat. Eine besondere Rolle in dieser Dissertation spielt ein von Varchenko in 1970ern entwickelter Algorithmus, durch den man die Abklingrate berechnen kann, wenn ϕ eine zweidimensionale reell-analytische Funktion ist.

In dieser Arbeit befassen wir uns mit dem Fourierrestriktionsproblem für Hyperflächen im $\mathbb{R}^3$ mit gemischten Lebesgueschen L^p-Normen. Genauer gesagt wollen wir bestimmen, für welche $p = (p_1, p_3) \in [1, 2]^2$ der Operator R zu einem stetigen Operator von $L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})$ nach $L^2(\rho \mathrm{d}S)$ fortgesetzt werden kann. Solche gemischten Lebesgueschen Normen sind von besonderer Bedeutung in der Theorie der partiellen Differentialgleichungen.

Im ersten Teil der Dissertation beweisen wir eine Verallgemeinerung eines jüngeren Ergebnisses von I. A. Ikromov und D. Müller auf gemischte Lebesguesche Normen. Ikromov und Müller haben lokale $L^p - L^2$-Fourierrestriktionsabschätzungen betrachtet, d. h., Abschätzungen, bei denen ρ in einer kleinen Umgebung von einem gegebenen Punkt getragen ist. Sie haben für eine große Familie von Hyperflächen im $\mathbb{R}^3$, die insbesondere alle reell-analytischen Hyperflächen einschließt, scharfe $L^p - L^2$-Fourierrestriktionsabschätzungen bewiesen. Wir beschränken uns in dieser Arbeit jedoch auf die Klasse derjenigen Hyperflächen, die man als Graphen von Funktionen ϕ darstellen kann, die entweder in adaptierten Koordinaten gegeben sind, oder die nicht in adaptierten Koordinaten gegeben sind, aber deren sogenannte lineare Höhe strikt kleiner als zwei ist. In unserem Beweis bauen wir auf den Methoden von Ikromov und Müller auf und kombinieren diese mit einer Methode aus einer Arbeit von J. Ginibre und G. Velo, die wichtig zur Gewinnung gemischter L^p-Abschätzungen ist.

Im zweiten Teil der Dissertation beweisen wir eine weitreichende Verallgemeinerung eines Ergebnisses aus einer jüngeren Arbeit von A. Carbery, C. E. Kenig und S. N. Ziesler zur Fourierrestriktion mit dämpfendem Gewichtsfaktor für Hyperflächen im $\mathbb{R}^3$, die man als Graphen von homogenen Polynomen darstellen kann. Carbery, Kenig und Ziesler haben globale Abschätzungen betrachtet, d.h., die Funktion ρ hat die Rolle einer Gewichtsfunktion gespielt, derer Träger sich auf die ganze Hyperfläche S erstreckt. Unser Zugang ist gegenüber dem von Carbery, Kenig und Ziesler, der Methoden aus der algebraischen Topologie sowie algebraischen Geometrie nutzt, deutlich elementarer und erlaubt zudem, erheblich allgemeinere Klassen von Hyperflächen zu behandeln, nämlich alle Hyperflächen, die man als Graphen gemischt-homogener Funktionen darstellen kann. Ein wesentlicher Schritt in unserem Zugang besteht in der Klassifizierung aller Singularitäten dieser Hyperflächen. Aufbauend darauf können wir anschließend ähnlich wie im ersten Teil der Dissertation die Erweiterungen der Methoden und Ideen von Ikromov und Müller und von Ginibre und Velo anwenden.

Als eine Anwendung der obigen Resultate leiten wir schließlich neue Strichartz-Abschätzungen für eine große Familie von partiellen Differentialoperatoren und Pseudodifferentialoperatoren mit Hilfe des Christ-Kiselev Lemmas her.

Danksagung

Ich würde gerne zuerst Herrn Prof. Dr. Detlef Müller für seine Betreuung und Unterstützung danken, und der Deutschen Forschungsgemeinschaft, ohne deren finanzielle Unterstützung im Rahmen des DFG-Projekts 237750060 meine Promotion nicht möglich gewesen wäre.

Također bih se htio zahvaliti doc. dr. sc. Vjekoslavu Kovaču za to što me je zainteresirao za područje harmonijske analize.

Ich danke weiterhin Alexander, Florian, Frederic, Hongwei, Javad, Lukas, Marcin, Marco, Markus, Mareike, Nico, Patrick und Stefan für eine sehr angenehme Zeit in Kiel und für die vielen gemeinsamen Erlebnisse. Ein besonderer Dank geht an Stefan für seine Hilfsbereitschaft sowie das Lesen einiger Teile der Dissertation.

Konačno, zahvalio bih se svojoj obitelji – Grguru, Klaudiji, Korneliji, roditeljima i ujaku – za podršku tijekom cijelog doktorskog studija.

Contents

Chapter 1

Introduction

1.1 The Fourier restriction problem

In the late 1960s E. M. Stein posed the following problem. Let us fix a hypersurface S in $\mathbb{R}^n$ and denote its Riemannian surface measure by $\mathrm{d}S$. If we choose on S a smooth compactly supported function $\rho \geqslant 0$, $\rho \in C_c^\infty(S)$, then one may ask for which $p, q \in [1, \infty]$ the estimate

$$\left(\int |\widehat{f}|^q \rho \mathrm{d}S \right)^{1/q} \leqslant \mathrm{C}_{p,q,\rho,S} \, \|f\|_{L^p(\mathbb{R}^n)}, \qquad f \in \mathcal{S}(\mathbb{R}^n), \tag{1.1.1}$$

holds true. Here $\mathcal{S}(\mathbb{R}^n)$ denotes the space of Schwartz functions on $\mathbb{R}^n$ and the Fourier transform $\widehat{f} = \mathcal{F} f$ is given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) \mathrm{d}x, \qquad \xi \in \mathbb{R}^n.$$

The inverse Fourier transform, denoted by $\check{f} = \mathcal{F}^{-1} f$, is then given by

$$\check{f}(x) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi) \mathrm{d}\xi, \qquad x \in \mathbb{R}^n.$$

Note that the a priori estimate (1.1.1) implies by density of $\mathcal{S}$ in L^p spaces, $1 \leqslant p < \infty$, that the restriction operator

$$R : f \mapsto \widehat{f}\,|_S$$

can be extended to an $L^p(\mathbb{R}^n, \mathrm{d}x) \to L^q(S, \rho\mathrm{d}S)$ bounded linear map.

Let us make some elementary observations. The classical Hausdorff-Young inequality states that the Fourier transform maps $L^p(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n)$ continuously for $1 \leqslant p \leqslant 2$, where $p' = p/(p-1)$ is the conjugate exponent. In fact, one has that $\mathcal{F} : L^1(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$, where $C_0(\mathbb{R}^n)$ denotes the space of continuous functions which vanish at infinity. On the other hand, if one normalizes the Fourier transform appropriately, then $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ becomes an isometric isomorphism. These observations imply that on the one hand the restriction operator R is always bounded as a map from L^1 to L^q for any

$q \in [1, \infty]$, and on the other hand it is never bounded as a map from L^2 to L^q for any $q \in [1, \infty]$, since the measure $\mathrm{d}S$ is supported on a set which has Lebesgue measure 0.

Further observations can be made by considering the adjoint operator

$$R^* : g \mapsto \mathcal{F}^{-1}(g\, \rho \mathrm{d}S),$$

usually called the extension operator, defined initially for, say, smooth functions g on S. It is bounded as a map from $L^{q'}(S, \rho\mathrm{d}S)$ to $L^{p'}(\mathbb{R}^n, \mathrm{d}x)$, that is

$$\| \mathcal{F}^{-1}(g\, \rho \mathrm{d}S)\|_{L^{p'}(\mathbb{R}^n, \mathrm{d}x)} \leqslant \mathrm{C}_{p,q,\rho,S} \, \|g\|_{L^{q'}(S,\rho\mathrm{d}S)}, \tag{1.1.2}$$

if and only if R is bounded as a map from $L^p(\mathbb{R}^n, \mathrm{d}x)$ to $L^q(S, \rho\mathrm{d}S)$. Since a bump function $\chi \in C_c^\infty(S)$ is in $L^{q'}(S, \rho\mathrm{d}S)$ for any q, one expects to obtain a necessary condition on p by considering $R^*(\chi) = \mathcal{F}^{-1}(\chi\rho\mathrm{d}S)$. By rotational and translational invariance of the Fourier transform we may assume that S is given as the graph of a function $\phi : \Omega \to \mathbb{R}$ satisfying $\phi(0) = 0$ and $\nabla\phi(0) = 0$, where Ω is an open set satisfying $0 \in \Omega \subseteq \mathbb{R}^{n-1}$. Thus

$$R^*(\chi)(x) = \int_{\mathbb{R}^{n-1}} e^{i(x'\cdot\xi' + x_n\phi(\xi'))} \chi(\xi', \phi(\xi'))\rho(\xi', \phi(\xi'))\sqrt{1 + |\nabla\phi(\xi')|^2}\mathrm{d}\xi',$$

where we split the coordinates $x = (x', x_n) \in \mathbb{R}^n$ with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. From this expression one can see that geometric properties such as curvature play an important role. Namely, if $\phi \equiv 0$, then the function $R^*(\chi)(x)$ is constant in x_n, and therefore can belong to only $L^\infty(\mathbb{R}^n)$. On the other hand, if we have nonvanishing Gaussian curvature, i.e., say $\phi(\xi') = |\xi'|^2$, then by using the so called van der Corput lemma in each coordinate one gets

$$|R^*(\chi)(x)| \leqslant C_\chi(1 + |x|)^{-\frac{n-1}{2}},$$

and so we see that $R^*(\chi) \in L^{p'}(\mathbb{R}^n)$ for any p satisfying $\frac{n-1}{2}p' > n$. Under the further conditions $\rho(0) \neq 0 \neq \chi(0)$ and that χ is supported in a sufficiently small neighbourhood of 0, one can see by means of the method of stationary phase that the condition $\frac{n-1}{2}p' > n$ is in fact necessary for (1.1.2) to hold.

Finally, if we plug $\chi_\varepsilon(\xi', \phi(\xi')) := \chi(\varepsilon^{-1}\xi', \phi(\varepsilon^{-1}\xi'))$ into (1.1.2) (this is the so called Knapp example) then by calculating the values on the left and right hand side of the inequality and considering the limit as $\varepsilon \to 0$ one obtains the necessary condition $\frac{1}{q} \geqslant \frac{n+1}{(n-1)p'}$. Thus, one is led to the following conjecture:

Conjecture 1. *Let S be a hypersurface in $\mathbb{R}^n$ with nonvanishing Gaussian curvature and let ρ be compactly supported in S. Then for any $p, q \in [1, \infty]$ satisfying $\frac{1}{p'} < \frac{n-1}{2n}$ and $\frac{1}{p'} \leqslant \frac{n-1}{n+1}\frac{1}{q}$ (see Figure 1.1) the operator R is bounded as a map from $L^p(\mathbb{R}^n, \mathrm{d}x)$ to $L^q(S, \rho\mathrm{d}S)$.*

The above conjecture has been and still is one of the central conjectures in modern harmonic analysis. It is closely related to many fundamental problems relating properties of the Fourier transform to the underlying geometric structures such as the problem of determining the Hausdorff dimension of Kakeya type sets, the problem of obtaining Strichartz

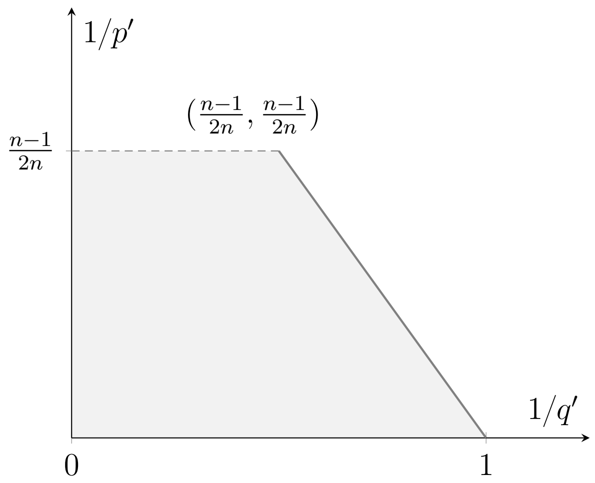

Figure 1.1: The conjectured range in which the restriction operator is bounded in the case when the hypersurface has nonvanishing Gaussian curvature.

and local smoothing estimates for PDEs, the boundedness properties of Bochner-Riesz means and averaging operators over hypersurfaces, and decoupling estimates (which in turn have deep consequences in number theory as most notably shown in the recent proof of the main conjecture in Vinogradov's mean value theorem by J. Bourgain, C. Demeter, and L. Guth [13]).

For $n = 2$ the Conjecture 1 was confirmed to hold in the early 1970s by C. Fefferman and E. M. Stein [34] and A. Zygmund [89] (see also the papers by L. Carleson and P. Sjölin [24] and L. Hörmander [46]). The first important result in the higher dimensional case was proved by P. A. Tomas and E. M. Stein [80] for the $(n-1)$-sphere:

Theorem 1.1.1. *Let S be the unit sphere in $\mathbb{R}^n$ and take $\rho \equiv 1$. Then the restriction operator R is bounded from $L^p(\mathbb{R}^n, \mathrm{d}x)$ to $L^2(S, \mathrm{d}S)$ for $1 \leqslant p \leqslant \frac{2n+2}{n+3}$.*

This result is sharp in the sense that when $q = 2$ is fixed, the given range for p is best possible.

Strikingly, the question whether the restriction operator R for the unit sphere is bounded in the full conjectured range (as given in Conjecture 1) remains open even in $\mathbb{R}^3$, and a lot of deep work has been done in this direction. A major impetus was given by J. Bourgain [11], [12] and T. Wolff [87] in the early 1990s followed by papers [64], [84], [65], [79] of A. Moyua, A. Vargas, L. Vega, and T. Tao, where the bilinear methods were developed and successfully applied. Subsequently multilinear methods were considered by J. Bennet, A. Carbery, and T. Tao [9] and J. Bourgain and L. Guth [14]. The most recent fundamental improvement was obtained by L. Guth [43], [44] who applied a new technique called polynomial partitioning.

Let us also mention some papers in the case of vanishing principal curvatures (such as the case of conical surfaces, and in particular the case of the light cone) done by B. Barcelo [6], [7] and later by A. Vargas, T. Wolff, and T. Tao [82], [83], [88], [78], as they were an important stepping stone in obtaining a better understanding of cases with nonvanishing Gaussian curvature. See also the more recent papers by S. Lee and A. Vargas [61] and S. Buschenhenke [16].

The case when one has both positive and negative principal curvatures (a prototypical example is the hyperbolic paraboloid in $\mathbb{R}^3$ determined by the equation $\xi_3 = \xi_1 \, \xi_2$) turned out to be more difficult compared to the case when all the principal curvatures have the same sign. First applications of the bilinear method to the case of the hyperbolic paraboloid were done by S. Lee and A. Vargas [60], [86], and some recent improvements of their result were obtained by C.-H. Cho and J. Lee [25], J. Kim [58], and B. Stovall [76]. Unlike the case when all principal curvatures are of the same sign, in this case new ideas are required when one considers small perturbations of a hypersurface. This opened a new line of research which has been recently pursued by S. Buschenhenke, D. Müller, and A. Vargas in their joint series of papers [17], [18], [19], [20].

In this thesis, however, we shall be interested exclusively in the Stein-Tomas $L^p \to L^2$ type estimates, with primary focus on hypersurfaces which have vanishing Gaussian curvature at the point $(0, \phi(0)) \in S$. As was already mentioned, Tomas and Stein proved the $L^p \to L^2$ estimate for the sphere. The main observations which render the $L^p \to L^2$ estimate significantly simpler compared to the general $L^p \to L^q$ estimates are that R is $L^p \to L^2$ bounded if and only if R^*R is $L^p \to L^{p'}$ bounded, and that the operator R^*R is in fact a convolution operator given by $f \mapsto f * \mathcal{F}^{-1}(\rho \mathrm{d}S)$. Thus, the boundedness of R is directly related to the behavior of the oscillatory integral

$$\mathcal{F}^{-1}(\rho \mathrm{d}S)(x) = \int_{\mathbb{R}^{n-1}} e^{i(x' \cdot \xi' + x_n \phi(\xi'))} \rho(\xi', \phi(\xi')) \sqrt{1 + |\nabla \phi(\xi')|^2} \mathrm{d}\xi'. \tag{1.1.3}$$

A useful result by A. Greenleaf [41] from 1980 tells us that if one can obtain a decay estimate on the Fourier transform of the measure $\rho \mathrm{d}S$ (i.e., an estimate on the above oscillatory integral) of the form

$$| \mathcal{F}^{-1}(\rho \mathrm{d}S)(x)| \leqslant C_{\rho,S} (1 + |x|)^{-1/h}, \qquad x \in \mathbb{R}^n,$$

for some $h > 0$, then the associated $L^p \to L^2$ restriction estimate holds true for $p' \geqslant 2(h+1)$. When the decay estimate is sharp, Greenleaf's result was shown to be optimal in the case of convex surfaces of finite type (see the papers by J. Bruna, A. Nagel, and S. Wainger [15] and A. Iosevich [52]). However, Greenleaf's result is not sharp in general. In a series of articles I. A. Ikromov and D. Müller [49], [50] (see also their work with M. Kempe [48]) made substantial progress in obtaining $L^p \to L^2$ restriction estimates for large classes of surfaces in $\mathbb{R}^3$, culminating in the proof of the following deep result of Ikromov and Müller [51]:

Theorem 1.1.2. *Let S, ρ, and ϕ be as above and assume that ϕ is a function of finite type and that it is linearly adapted in its original coordinates. Then the estimate* (1.1.1) *holds true for all ρ with support contained in a sufficiently small neighbourhood of* 0 *when $q = 2$ and when either*

(a) ϕ is adapted in its original coordinates and $p \geqslant 2(h(\phi) + 1)$, or

(b) ϕ is not adapted in its original coordinates, satisfies the Condition (R), and $p \geqslant 2(h^{res}(\phi) + 1)$.

The above result is sharp as can be shown by using a Knapp-type example. The assumption of linear adaptedness does not reduce generality, and the assumption that ϕ

is of finite type means simply that the Taylor series of ϕ at 0 does not vanish identically. The quantities $h(\phi)$ and $h^{\text{res}}(\phi)$ are respectively the height and the restriction height of the function ϕ. We recall the definitions of these quantities in the next section. Condition (R) is a factorization condition satisfied by real analytic function, but not for smooth functions in general. It remains unkown whether the above theorem holds true in absence of this condition. Thus, we see that Ikromov and Müller have in particular solved the (local) $L^p - L^2$ Fourier restriction problem for all analytic surfaces.

The techniques developed in the proof of Theorem 1.1.2 build upon the work of A. N. Varchenko [85] (and more generally Arnold's school which developed singularity theory and noted the importance of Newton polyhedra in estimating oscillatory integrals [3], [4]) and upon the more recent results of D. H. Phong, E. M. Stein, and J. A. Sturm [69], [70], where they additionally applied Puiseux series expansions to estimate oscillatory integrals. Newton polyhedra turned out to be of fundamental importance in understanding asymptotics of oscillatory integrals of the form

$$\int_{\mathbb{R}^n} e^{i\lambda\phi(x)} a(x) \mathrm{d}x,$$

where a is a smooth function supported in a sufficiently small neighbourhood of the origin. It was shown by I. N. Bernstein and S. I. Gelfand [10] and M. Atiyah [5] that for real analytic functions ϕ there exists an asymptotic expansion in λ of the above oscillatory integral with terms of the form $\lambda^{-r}(\log\lambda)^j$ for r a positive rational number and $0 \leqslant j \leqslant n-1$. These proofs are based on a resolution of singularities result of H. Hironaka [45]. However, for a concrete function ϕ it was not clear from their proofs how to calculate even the main term in the asymptotic expansion. For the case of two-dimensional real analytic functions this was resloved by A. N. Varchenko [85] who found an algorithm on how to obtain the exponent of the main term of the asymptotic expansion (and whether the logarithmic factor appears). More precisely, he constructed a so called adapted coordinate system in which the Newton distance $d(\phi)$ of the Newton polyhedron of ϕ is maximal – the maximal distance is called the Newton height $h(\phi)$, and one can show that the decay of the above oscillatory integral is $\lambda^{-1/h(\phi)}$ (up to a logarithmic factor). It was later shown by V. N. Karpushkin [55] that Varchenko's result is stable under perturbations, which means we automatically obtain a decay estimate on the Fourier transform in (1.1.3) (this estimate can be interpreted as a uniform estimate over linear perturbations of the phase function ϕ). These result were generalized by Ikromov and Müller to all two-dimensional smooth functions of finite type [49], [50]. Let us mention here that for higher dimensions a resolution of singularities algorithm was recently applied by T. C. Collins, A. Greenleaf, and M. Pramanik [27] which allowed them to determine the growth rate of sublevel sets.

The first main result of this thesis is a generalization of Theorem 1.1.2 to mixed norm Lebesgue spaces (though our result is restricted to the adapted case, and to the non-adapted case when the so called linear height is strictly below 2). More precisely, in the estimate (1.1.1) we shall consider $L^p(\mathbb{R}^3)$ as the space of functions satisfying

$$\left(\int \left(\int\int |f|^{p_1}(x_1, x_2, x_3) \mathrm{d}x_1 \mathrm{d}x_2 \right)^{p_3/p_1} \mathrm{d}x_3 \right)^{1/p_3} < \infty,$$

i.e., we only differentiate between the tangential and the normal direction to the surface S at the point $0 \in S$. In this case p denotes the pair $(p_1, p_3) \in [1,2]^2$. Estimates for these kind of mixed norms are of particular importance in PDE theory. We shall state our result in Section 1.3 below. It has been published in [67] and its proof is contained in Chapter 3.

The second problem we shall be dealing with in this thesis is concerned with what happens for global surfaces S of the form

$$\Big\{(x_1, x_2, \phi(x_1, x_2)) \in \mathbb{R}^3 | (x_1, x_2) \in \mathbb{R}^2 \backslash \{0\}\Big\},$$

where the function $\phi : \mathbb{R}^2 \backslash \{0\} \to \mathbb{R}$ is smooth and mixed homogeneous, i.e., it satisfies

$$\phi(r^{\kappa_1} x_1, r^{\kappa_2} x_2) = r^{\mathcal{D}} \phi(x_1, x_2), \qquad r > 0,$$

for some "homogeneity" weights $(\kappa_1, \kappa_2) \in (0,\infty)^2$ and $\mathcal{D} \in \{-1, 0, 1\}$.

Mixed homogeneous surfaces have been studied in numerous problems in harmonic analysis, often as model cases (see for example [53], [36], [47], [37], [22], [40], [30], [71]). Our motivation, however, is that for mixed homogeneous surfaces one can obtain global results, and as a consequence they can be applied in PDE theory. Namely, we shall prove the estimate (1.1.1) for functions ρ which are mixed homogeneous weights satisfying $\rho \neq 0$ almost everywhere on S. We shall consider two types of weights. The first type is a weight which is $\neq 0$ everywhere on S, and the second type of weight is a weight which has a so called mitigating (or damping) factor of the form

$$|\mathcal{H}_\phi|^{\mathfrak{s}},$$

for some $\mathfrak{s} > 0$, where $\mathcal{H}_\phi$ denotes the Hessian determinant of ϕ. Note that this weight has roots at the degenerate points of ϕ, and thus the (local) Fourier restriction estimate holds for a wider range of exponents compared to when the first type of weight is used.

The first use of mitigating factors goes back to P. Sjölin [73] in the early 1970s, and they were later considered in the papers [28], [32], [57] in the early 1990s. Interestingly, when one uses mitigating factors one can get results even for flat surfaces [23], [1], [21]. A particularly general (though weak type) estimate was shown by D. M. Oberlin [66] for surfaces having a bounded generic multiplicity, and P. T. Gressman [42] obtained recently decay estimates for oscillatory integrals with damping factors for a certain class of singularities.

In this thesis we shall give a wide generalization of a more recent result of A. Carbery, C. E. Kenig, and S. N. Ziesler [22] where they proved the following Fourier restriction estimate with a damping factor for "isotropically" homogeneous (i.e., when $\kappa_1 = \kappa_2$) polynomials ϕ:

$$\left(\int_{\mathbb{R}^2} |\widehat{f} \circ \phi|^2 |\mathcal{H}_\phi|^{1/4} \mathrm{d}\xi \right)^{1/2} \leqslant C_\phi \, \|f\|_{L^{4/3}(\mathbb{R}^3)}, \qquad f \in \mathcal{S}(\mathbb{R}^3).$$

The methods we use turn out to be considerably more elementary and build on techniques developed by Ikromov and Müller in [51] and by J. Ginibre and G. Velo in [38]. We state our results (available as a preprint [68]) more precisely in Section 1.4, and prove them in Chapter 4. Additionally, in Section 1.4 we state the precise Strichartz estimate [77] which one obtains as a consequence of our Fourier restriction estimates.

1.2 Newton polyhedron

In this section we review fundamental concepts related to Newton polyhedra which are key to understanding the proof of Theorem 1.1.2.

Let the surface S be given as the graph $S = S_\phi := \{(x_1, x_2, \phi(x_1, x_2)) : x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2\}$ of a smooth and real-valued function ϕ defined on an open neighbourhood Ω of the origin. We can assume without loss of generality that $\phi(0) = 0$, and we take Ω to be a sufficiently small neighbourhood of the origin in $\mathbb{R}^2$. In the mixed norm case we cannot use the rotational invariance of the Fourier transform in order to reduce to the case $\nabla\phi(0) = 0$. Instead we can use a different linear transformation (for details see Section 2.1), and so we may and shall assume $\nabla\phi(0) = 0$.

Next, we impose on ϕ to be a function of *finite type* at 0. This means that there exists a multi-index $\alpha \in \mathbb{N}_0^2$ such that $\partial^\alpha\phi(0) \neq 0$. By continuity, ϕ is of finite type on a neighbourhood of 0. We may therefore assume that ϕ is of finite type at each point of Ω. We define the *Taylor support* of ϕ as the set

$$\mathcal{T}(\phi) := \Big\{\alpha \in \mathbb{N}_0^2 : \partial^\alpha\phi(0) \neq 0\Big\}.$$

The *Newton polyhedron* $\mathcal{N}(\phi)$ of ϕ is the convex hull of the set

$$\bigcup_\alpha \Big\{(t_1, t_2) \in \mathbb{R}^2 : t_1 \geqslant \alpha_1, t_2 \geqslant \alpha_2\Big\},$$

where the union is over all α such that $\partial^\alpha\phi(0) \neq 0$ (and so $|\alpha| \geqslant 2$). See Figure 1.2. Both, edges and vertices, are called *faces* of $\mathcal{N}(\phi)$. We define the *Newton diagram* $\mathcal{N}_d(\phi)$ of ϕ to be the union of all *compact* faces of $\mathcal{N}(\phi)$.

If we are given a face e_0 of $\mathcal{N}(\phi)$, we can define its associated (formal) series

$$\phi_{e_0}(x_1, x_2) := \sum_{\alpha \in e_0 \cap \mathcal{T}(\phi)} \frac{\partial^\alpha\phi(0)}{\alpha!} x^\alpha. \tag{1.2.1}$$

If e_0 is a compact face, then $\phi_{e_0}(x_1, x_2)$ is a mixed homogeneous polynomial. This means that there exists a weight $\kappa^{e_0} = (\kappa_1^{e_0}, \kappa_2^{e_0}) \in [0, \infty)^2$ such that for any $r > 0$ we have

$$\phi_{e_0}(r^{\kappa_1^{e_0}} x_1, r^{\kappa_2^{e_0}} x_2) = r\phi_{e_0}(x_1, x_2),$$

and we call ϕ_{e_0} a κ^{e_0}-homogeneous polynomial. κ^{e_0} is uniquely determined if and only if e_0 is not a vertex. In fact, in the case when e_0 is an edge, we define $L_{\kappa^{e_0}}$ to be the unique line containing e_0:

$$e_0 \subseteq L_{\kappa^{e_0}}. \tag{1.2.2}$$

Then the weight κ^{e_0} is uniquely determined by the relation

$$L_{\kappa^{e_0}} = \Big\{(t_1, t_2) \in \mathbb{R}^2 : \kappa_1^{e_0} t_1 + \kappa_2^{e_0} t_2 = 1\Big\}. \tag{1.2.3}$$

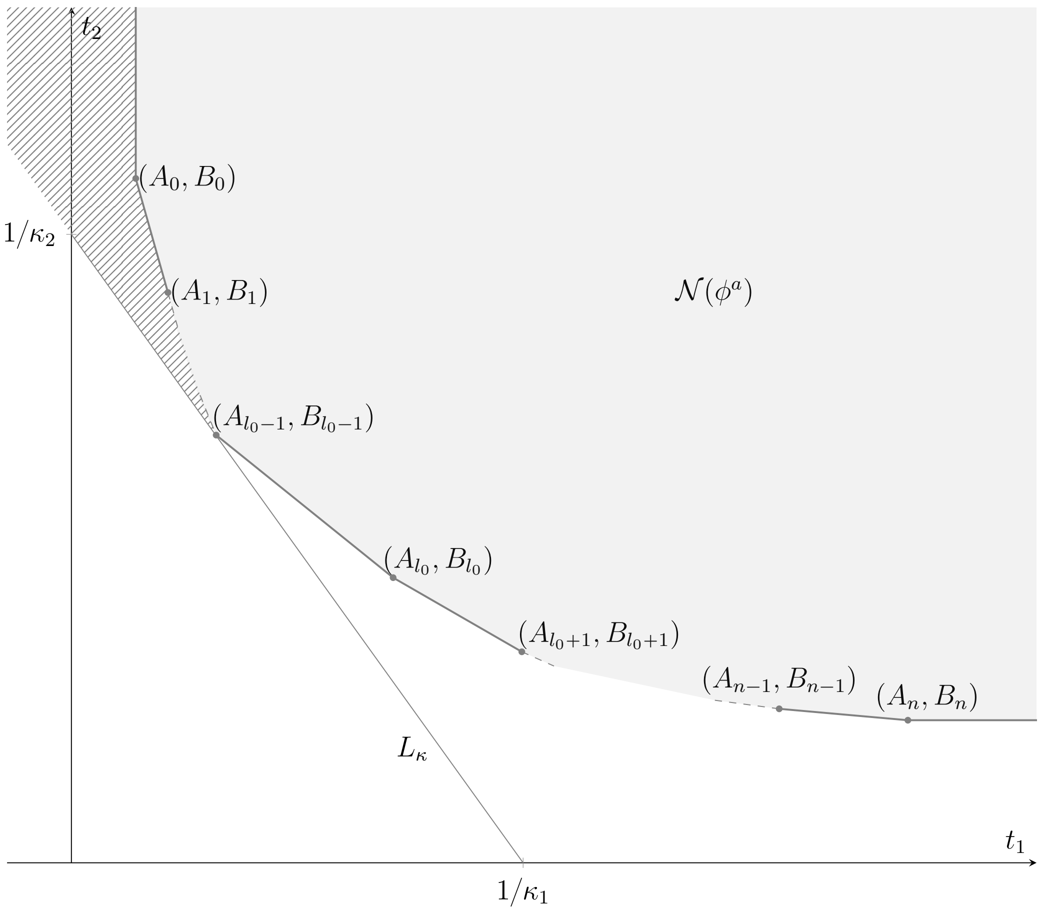

Figure 1.2: The (augmented) Newton polyhedron associated to ϕ^a.

When e_0 is an unbounded face, $\phi_{e_0}(x_1, x_2)$ is to be taken only as a formal power series. Note that then e_0 is either a vertical or horizontal edge of $\mathcal{N}(\phi)$, and we can also find unique $\kappa_1^{e_0}$ and $\kappa_2^{e_0}$ (one of them being 0 in this case) such that (1.2.2) holds.

Of particular interest is the *principal face* $\pi(\phi)$ defined as the face of minimal dimension of $\mathcal{N}(\phi)$ which intersects the bisectrix $\{(t_1, t_2) \in \mathbb{R}^2 : t_1 = t_2\}$. Its associated series (or homogeneous polynomial) is called the *principal part* of ϕ and denoted by $\phi_{\text{pr}} := \phi_{\pi(\phi)}$. Let the weight $\kappa = (\kappa_1, \kappa_2)$ determine the line L_κ as in (1.2.3) containing the principal face of $\mathcal{N}(\phi)$ if it is an edge, or when it is a vertex, let κ determine the edge of $\mathcal{N}(\phi)$ having the principal face as its left endpoint. Interchanging the x_1 and x_2 coordinates, if necessary, we may always assume that

$$\kappa_2 \geqslant \kappa_1.$$

We define $|\kappa| := \kappa_1 + \kappa_2$ and denote the ratio κ_2/κ_1 by m so that $m \geqslant 1$.

The *Newton distance* $d(\phi)$ of ϕ is defined to be the coordinate d of the point (d, d) which is the intersection of the bisectrix and the principal face of $\mathcal{N}(\phi)$. One can easily see that if $\kappa = (\kappa_1, \kappa_2)$ determines the line containing the principal face (or any of the supporting lines to $\mathcal{N}(\phi)$ in case $\pi(\phi) = \{(d,d)\}$), then we have

$$d(\phi) = \frac{1}{\kappa_1 + \kappa_2}.$$

The *Newton height* $h(\phi)$ of ϕ is defined as

$$h(\phi) = \sup\{d(\phi \circ \varphi) : \varphi \text{ a smooth local coordinate change}\}.$$

By a smooth local coordinate change we mean a function φ which is smooth and invertible in a neighbourhood of the origin, and $\varphi(0) = 0$. We also define the *linear height* as

$$h_{\text{lin}}(\phi) = \sup\{d(\phi \circ \varphi) : \varphi \text{ a linear coordinate change}\}.$$

For a coordinate change φ we shall denote the new coordinates by $y = \varphi(x)$. In this case we also write $d_y = d(\phi \circ \varphi)$. We say that ϕ is *adapted* in the y coordinates if $d_y = h(\phi)$. Analogously, we say that ϕ is *linearly adapted* in coordinates y if $d_y = h_{\text{lin}}(\phi)$. When ϕ is adapted in its original coordinates x we say that ϕ is adapted, and if ϕ is not adapted in its original coordinates, then we say that ϕ is non-adapted. Analogous expressions we shall use for linear adaptedness. We obviously always have

$$d_x = d(\phi) \leqslant h_{\text{lin}}(\phi) \leqslant h(\phi).$$

The existence of an adapted coordinate system for real analytic functions on $\mathbb{R}^2$ was first proven by Varchenko in [85]. He gave an explicit algorithm on how to construct an adapted coordinate system. His result was generalized in [49] where it was shown that an adapted coordinate system exists for general smooth functions of finite type. It turns out that in the smooth case one can also essentially use Varchenko's algorithm. In this thesis when we refer to Varchenko's algorithm we shall always mean the variant used in [49]. In this variant one constructs an adapted coordinate system in the form of a nonlinear shear transformation

$$y_1 = x_1, \quad y_2 = x_2 - \psi(x_1).$$

The smooth real-valued function ψ can be taken in the real-analytic case to be the *principal root jet* of ϕ as defined in [51]. We denote the function ϕ in the new (adapted) coordinates by ϕ^a. Then we have

$$\phi^a(y) = \phi(y_1, y_2 + \psi(y_1)).$$

We remark that when ϕ is not adapted, then $m = \kappa_2/\kappa_1$ is a positive integer and $\psi(x_1) - b_1 x_1^m = \mathcal{O}(x_1^{m+1})$ for some nonzero real constant b_1.

We recall the definition of *Varchenko's exponent* $\nu(\phi) \in \{0,1\}$ next. If $h(\phi) \geqslant 2$ and there exists an adapted coordinate system y such that in these coordinates the principal face of $\phi^a(y)$ is a vertex, we define $\nu(\phi) := 1$. In all other cases we take $\nu(\phi) := 0$. In particular $\nu(\phi) = 0$ whenever $h(\phi) < 2$. A concrete characterisation for determining when an adapted coordinate system having the principal face as a vertex exists can be found in [50, Lemma 1.5].

Let us discuss next linear adaptedness. We assume that $h_{\mathrm{lin}}(\phi) < h(\phi)$, i.e., that we cannot achieve adapted coordinates with a linear coordinate change. In [51, Section 1.3] it was shown that in this case we can always find a linearly adapted coordinate system, and [51, Proposition 1.7] gives an explicit characterisation of when a coordinate system is linearly adapted. It was shown in particular that if the coordinate system x is not already linearly adapted, then one just needs to apply the first step of Varchenko's algorithm in order to obtain it.

Since in our mixed norm case we consider only $p_1 = p_2$, we can freely use linear coordinate changes in "tangential" variables (x_1, x_2) in the expression (1.3.1). Thus we may assume without loss of generality that either the original coordinate system x is already adapted, or that it is at least linearly adapted. In particular, we may assume $d(\phi) = h_{\mathrm{lin}}(\phi)$.

The final important concept we recall is the *augmented Newton polyhedron* $\mathcal{N}^{\mathrm{res}}(\phi^a)$ of a non-adapted ϕ (note the slight change in notation compared to [51], where $\mathcal{N}^{\mathrm{r}}(\phi^a)$ is used instead). $\mathcal{N}^{\mathrm{res}}(\phi^a)$ is defined as the convex hull of the set

$$\mathcal{N}(\phi^a) \cup L^+,$$

where L^+ is defined as follows. Let L_κ be the line containing the principal face $\pi(\phi)$ of $\mathcal{N}(\phi)$ and let $P = (t_1^P, t_2^P)$ be the point on $L_\kappa \cap \mathcal{N}(\phi^a)$ with the smallest t_2 coordinate. Such a point always exists. Then L^+ is the ray

$$\Big\{(t_1, t_2) \in L_\kappa : t_2 \geqslant t_2^P\Big\}.$$

(See Figure 1.2).

1.3 Local mixed norm Fourier restriction results

As in the previous section, let the surface S be given as the graph

$$S = S_\phi := \Big\{(x_1, x_2, \phi(x_1, x_2)) : x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2\Big\}$$

of a smooth and real-valued function ϕ defined on an open neighbourhood Ω of the origin. Recall that $\mathrm{d}S$ denotes the surface measure of S. In this section we shall discuss the Fourier restriction estimate

$$\left(\int |\widehat{f}|^2 \rho \mathrm{d}S \right)^{1/2} \leqslant \mathrm{C} \, \|f\|_{L^p(\mathbb{R}^3)} = \mathrm{C}(p, \rho, \phi) \|f\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})}, \qquad f \in \mathcal{S}(\mathbb{R}^3), \tag{1.3.1}$$

and give a precise statement of our results. Namely, we state Theorem 1.3.1 which gives the necessary conditions, and Theorem 1.3.2 which gives us the mixed norm Fourier restriction estimates in the adapted case and the case $h_{\mathrm{lin}}(\phi) < 2$. We recall that $\rho \in C_c^\infty(S)$ is supported in a sufficiently small neighbourhood of the origin.

Let us briefly review all the conditions on the function ϕ which we may assume without loss of generality when considering the mixed norm restriction problem:

- $\phi(0) = 0$ and $\nabla\phi(0) = 0$,
- ϕ is of finite type on Ω,
- the weight κ determined by the principal face of $\mathcal{N}(\phi)$ (or by the edge containing the principal face as its left endpoint) satisfies $m = \kappa_2/\kappa_1 \geqslant 1$, and
- the original coordinate system x is either adapted, or linearly adapted but not adapted. In both cases we have $d(\phi) = h_{\text{lin}}(\phi)$.

We begin by stating necessary conditions which will be obtained by means of Knapp-type examples. When ϕ is not adapted we denote by

$$K : [0, \kappa_1] \to [0, +\infty]$$

the function defined in the following way. Consider all the lines of the form

$$L_{\tilde{\kappa}} = \Big\{(t_1, t_2) \in \mathbb{R}^2 : \tilde{\kappa}_1 t_1 + \tilde{\kappa}_2 t_2 = 1\Big\}, \tag{1.3.2}$$

where $\tilde{\kappa} \in [0, \infty)^2$ is a weight. For each $0 \leqslant \tilde{\kappa}_1 \leqslant \kappa_1$ there is a unique $\tilde{\kappa}_2$ so that (1.3.2) determines a supporting line $L_{\tilde{\kappa}}$ to $\mathcal{N}^{\text{res}}(\phi^a)$. We then define $K(\tilde{\kappa}_1)$ to be $\tilde{\kappa}_2$ for the given $\tilde{\kappa}_1 \in [0, \kappa_1]$ (see Figure 1.3). Note that then the weight $(0, K(0))$ determines the line containing the horizontal edge of the augmented Newton polyhedron, i.e., the right most edge of $\mathcal{N}^{\text{res}}(\phi^a)$. The weight $(\kappa_1, K(\kappa_1)) = \kappa$ determines the line containing the edge associated to the principal face of $\mathcal{N}(\phi)$ which is the left most edge of $\mathcal{N}^{\text{res}}(\phi^a)$.

Denote by $\mathcal{L}$ the *Legendre transformation* for a real-valued convex function K:

$$\mathcal{L}(K)[w] := \sup_{u \in [0, \kappa_1]} (wu - K(u)).$$

Then we may state the necessary conditions in the following way:

Theorem 1.3.1. *Let ϕ be as above and let us assume that the estimate* (1.3.1) *holds true with $\rho(0) \neq 0$. If ϕ is adapted, then we have the necessary condition*

$$\frac{1}{d(\phi) p_1'} + \frac{1}{p_3'} \leqslant \frac{1}{2d(\phi)}.$$

If K is as above and ϕ is linearly adapted, but not adapted, then we necessarily have

$$\frac{1}{p_3'} \leqslant -\frac{1}{2}\mathcal{L}(K)\Big[\frac{2 + 2m}{p_1'} - 1\Big].$$

Recall that $d(\phi) = h(\phi)$ when ϕ is adapted. The above theorem is a direct consequence of Proposition 3.1.1 in Section 3.1 below and the discussion in Subsection 3.1.2. The necessary conditions are depicted in Figure 1.4.

The first major result of this thesis is:

Theorem 1.3.2. *Let ϕ be as above and ρ supported in a sufficiently small neighbourhood of* 0. *If either*

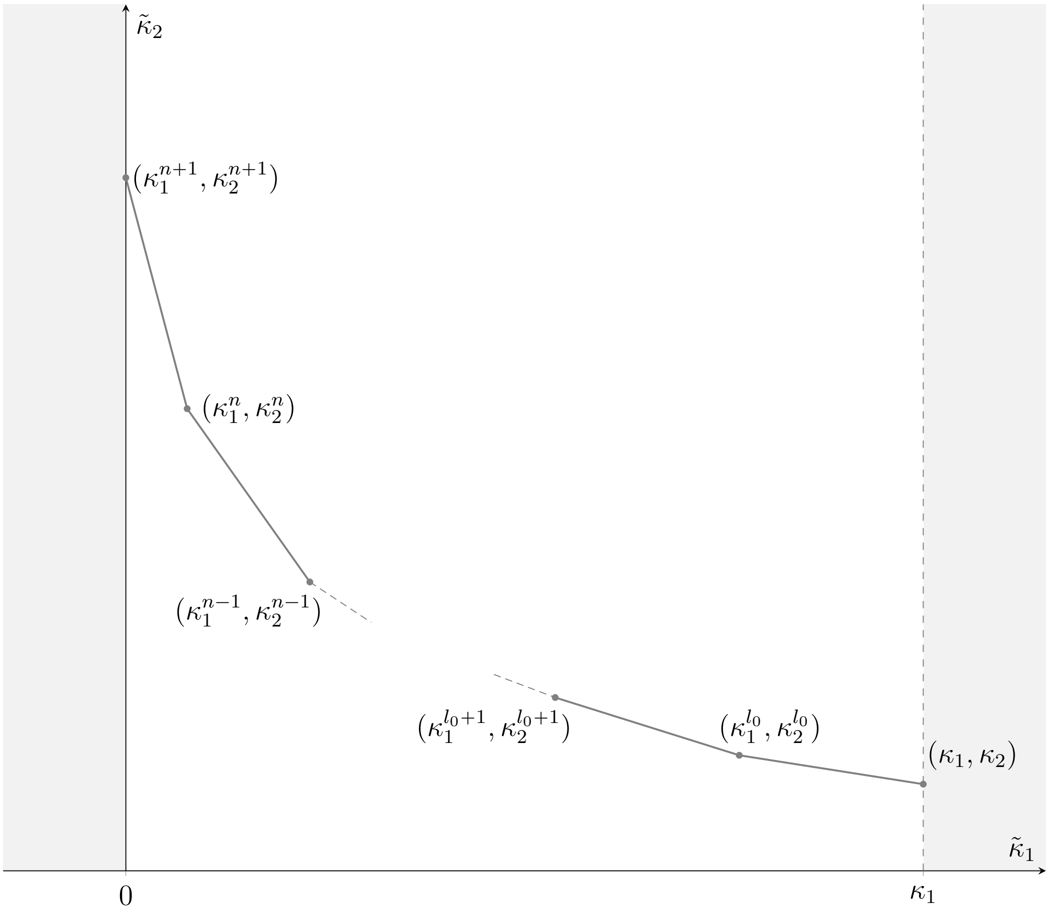

Figure 1.3: The typical form of the graph of the function $K : \tilde{\kappa}_1 \mapsto \tilde{\kappa}_2$.

(a) ϕ is adapted in its original coordinates, or

(b) ϕ is non-adapted, $h_{lin}(\phi) < 2$, and ϕ is real analytic,

then the estimate (1.3.1) *holds true for all* $(1/p_1', 1/p_3')$ *as determined by the necessary conditions stated in Theorem 1.3.1, except for the point* $(1/p_1', 1/p_3') = (0, 1/(2h(\phi)))$ *where it holds true if* $h(\phi) > 1$ *and* $\nu(\phi) = 0$, *but does not hold if* $\rho(0) \neq 0$ *and either* $h(\phi) = 1$ *or* $\nu(\phi) = 1$.

In case (b) we shall actually prove the claim for a more general class of functions than is stated here. The part (a) of the above theorem follows from Proposition 3.2.2, and the part (b) follows from Theorem 3.3.1. Let us mention that in the case $h_{\mathrm{lin}}(\phi) < 2$ it turns out that we always have $\nu(\phi) = 0$, which will be important for the boundary point $(1/p_1', 1/p_3') = (0, 1/(2h(\phi)))$.

In this thesis we do not deal with the non-adapted case when $h_{\mathrm{lin}}(\phi) \geqslant 2$ in its full generality. Let us briefly comment how one can easily get some preliminary Fourier restriction estimates. Namely, the abstract result from [56] by M. Keel and T. Tao implies that we automatically have the Fourier restriction estimate for the region labeled by KT in Figure 1.4 below. For details we refer to Proposition 3.2.1. One can combine this result with the case $p_1 = p_3$ from Theorem 1.1.2 and get by interpolation the region labeled by IM in Figure 1.4.

For the proof of Theorem 1.3.2 (or more precisely, for the proof of Proposition 3.2.2 and Theorem 3.3.1, which are to be found in Chapter 3) we shall appropriately modify the techniques from [51] and use essentially the same phase space decompositions as in [51]. The main additional ingredients we shall use are the ideas from [38] (see also [56]) for handling mixed norms. In our case additional complications appear which were absent in the corresponding cases in [51] and some of which resemble problems appearing in some of the final chapters of [51]. For example, after making a phase space decomposition of the kernel of the convolution operator obtained by the "R^*R technique", a recurring theme will be that we will not be able to sum absolutely the operators associated to the decomposition pieces, in contrast to [51] where these operators were absolutely summable. A further interesting feature of the mixed norm case is that certain estimates for the mixed norm endpoint become invariant under the scalings considered in [51].

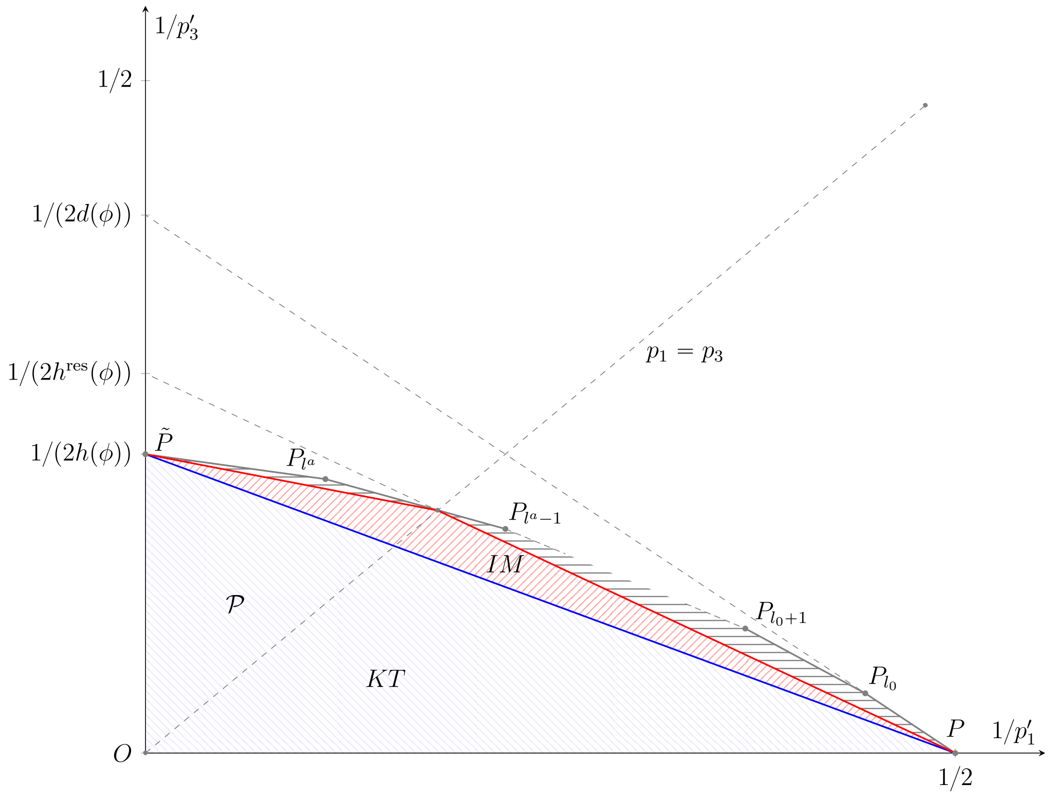

Figure 1.4: Necessary conditions in the $(1/p_1', 1/p_3')$-plane.

1.4 Global Fourier restriction results and application to PDEs

Let us fix a weight $\kappa = (\kappa_1, \kappa_2) \in (0, \infty)^2$, recall that $|\kappa| := \kappa_1 + \kappa_2$, and introduce its associated κ-mixed homogeneous dilations in $\mathbb{R}^2$ by

$$\delta_r(x_1, x_2) = (r^{\kappa_1} x_1, r^{\kappa_2} x_2), \qquad r > 0.$$

In this section we state Fourier restriction results for mixed homogeneous surfaces S, i.e., surfaces given as graphs of smooth functions $\phi : \mathbb{R}^2\backslash\{0\} \to \mathbb{R}$ which are κ-mixed homogeneous of degree $\mathcal{D}$:

$$\phi \circ \delta_r(x_1, x_2) = r^{\mathcal{D}} \phi(x_1, x_2), \qquad r > 0.$$

We may and shall assume without loss of generality that $\mathcal{D} \in \{-1, 0, 1\}$. Note that when $\mathcal{D} = -1$ the function ϕ blows up at the origin. The mixed norm estimate we are interested in is

$$\|\widehat{f}\|_{L^2(\mathrm{d}\mu)} \leqslant C\|f\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})}, \qquad f \in \mathcal{S}(\mathbb{R}^3), \tag{1.4.1}$$

where μ is the surface carried measure

$$\langle \mu, f \rangle = \int_{\mathbb{R}^2\backslash\{0\}} f(x_1, x_2, \phi(x_1, x_2))\, \mathcal{W}(x_1, x_2)\, \mathrm{d}x \tag{1.4.2}$$

and $p = (p_1, p_3) \in (1, 2)^2$. The weight $\mathcal{W} \geqslant 0$ is added in order to insure that the measure has a scaling invariance which will enable us to reduce global estimates to local ones by a Littlewood-Paley argument. We take $\mathcal{W}$ to be κ-homogeneous of degree $\mathcal{D}_{\mathcal{W}}$ and consider two particular cases. The function $\mathcal{W}$ will be either equal to

$$|x|_\kappa^{\mathcal{D}_{\mathcal{W}}} = (|x_1|^{1/\kappa_1} + |x_2|^{1/\kappa_2})^{\mathcal{D}_{\mathcal{W}}}, \tag{1.4.3}$$

or equal to the Hessian determinant of ϕ (denoted by $\mathcal{H}_\phi$) raised to the power $|\cdot|^{\mathfrak{s}}$, $\mathfrak{s} \in [0, 1/2)$, i.e.,

$$|\mathcal{H}_\phi(x)|^{\mathfrak{s}} = \left| \det \begin{bmatrix} \partial_{x_1}^2 \phi & \partial_{x_1}\partial_{x_2}\phi \\ \partial_{x_1}\partial_{x_2}\phi & \partial_{x_2}^2 \phi \end{bmatrix} \right|^{\mathfrak{s}}. \tag{1.4.4}$$

One can easily show that the Hessian determinant of ϕ is κ-mixed homogeneous of degree $2(\mathcal{D} - |\kappa|)$, and so in the case when $\mathcal{W}$ equals (1.4.4) the relation between $\mathcal{D}_{\mathcal{W}}$ and $\mathfrak{s}$ is $\mathcal{D}_{\mathcal{W}} = 2\mathfrak{s}(\mathcal{D} - |\kappa|)$. We shall determined $\mathcal{D}_{\mathcal{W}}$ in Chapter 4 so that the Fourier restriction estimate for μ is invariant under scaling. This choice depends in general on $p = (p_1, p_3)$.

In order to state our results (namely, Theorem 1.4.1, Theorem 1.4.2, Proposition 1.4.4, and Corollary 1.4.5) we first introduce the following two conditions on ϕ:

(H1) At any given point $(x_1, x_2) \neq (0, 0)$ where the Hessian determinant of ϕ vanishes at least one of the mappings $t \mapsto \partial_1^2\phi(t, x_2)$ or $t \mapsto \partial_2^2\phi(x_1, t)$ is of finite type at $t = x_1$ (resp. $t = x_2$).

(H2) The Hessian determinant $\mathcal{H}_\phi$ is not flat at any point $x \neq 0$.

It actually suffices to check the conditions only at points (x_1, x_2) in, say, the unit circle $\mathbb{S}^1$, by homogeneity. Furthermore, we remark that the condition (H2) is stronger than the condition (H1) (this follows from the calculations in Subsection 4.2.2 below).

Let us now introduce a further condition and two new quantities. For a point $v \in \mathbb{R}^2\backslash\{0\}$ let us define the function

$$\phi_v(x) := \phi(x+v) - \phi(v) - x \cdot \nabla\phi(v).$$

Note that $\phi_v(0) = 0$ and $\nabla\phi_v(0) = 0$. Then we shall often consider whether the following condition is satisfied at v:

(LA) There is a linear coordinate change which is adapted to ϕ_v at the origin.

The negation of this condition is just (NLA) in [51, Section 1.2].

Let us furthermore denote the linear height of ϕ_v by $h_{\mathrm{lin}}(\phi, v)$ and its Newton height by $h(\phi, v)$. We define the *global linear height* $h^{\mathrm{gl}}_{\mathrm{lin}}(\phi)$ and the *global Newton height* $h^{\mathrm{gl}}(\phi)$ by the respective expressions

$$h^{\mathrm{gl}}_{\mathrm{lin}}(\phi) = \sup_{v \in \mathbb{S}^1} h_{\mathrm{lin}}(\phi, v), \qquad h^{\mathrm{gl}}(\phi) = \sup_{v \in \mathbb{S}^1} h(\phi, v). \tag{1.4.5}$$

It will be clear from Section 4.2 that $h_{\mathrm{lin}}(\phi, v)$ and $h(\phi, v)$ do not change along the *homogeneity curve through* v defined as the curve

$$r \mapsto (r^{\kappa_1} v_1, r^{\kappa_2} v_2), \qquad r > 0,$$

and therefore in the above definitions of global linear height and global Newton height one could have taken the supremum over the set $\mathbb{R}^2\backslash\{0\}$ as well.

Theorem 1.4.1. *Let ϕ be mixed homogeneous satisfying condition (H2). Let μ be the measure defined as in* (1.4.2) *with $\mathcal{W}(x) = |\mathcal{H}_\phi(x)|^{\mathfrak{s}}$ for some fixed $\mathfrak{s} \geqslant 0$. If $\mathfrak{s} \in [0, \frac{1}{3}]$, then the Fourier restriction estimate* (1.4.1) *holds true for*

$$\Big(\frac{1}{p_1'}, \frac{1}{p_3'}\Big) = \Big(\frac{1}{2} - \mathfrak{s}, \mathfrak{s}\Big).$$

If (LA) is satisfied at all points $v \neq 0$, then the estimate holds true even if $\mathfrak{s} \in [0, \frac{1}{2})$. In particular, if $\kappa_1 = \kappa_2$, then (LA) is satisfied at all points $v \neq 0$ and the estimate holds true for any $\mathfrak{s} \in [0, \frac{1}{2})$.

Several comments are in order. Firstly, precise conditions for when the (LA) condition is satisfied at $v \neq 0$ can be checked by using the normal form tables in Section 4.2 (note that in the Proposition 1.4.4 below, where the normal forms are listed, only the Normal form (vi) is not in adapted coordinates). That one is restricted to $0 \leqslant \mathfrak{s} \leqslant 1/3$ in the case when (LA) is not satisfied is a consequence of a Knapp-type example, as we shall show in Subsubsection 4.3.6.

Secondly, in the case when $\mathcal{D} = 1 = |\kappa|$, one can extend the above estimate to the range where

$$\frac{1}{p_1'} + \frac{1}{p_3'} = \frac{1}{2}, \qquad \frac{1}{p_3'} \leqslant \mathfrak{s}.$$

The reason for this is that $\mathcal{D} = 1 = |\kappa|$ implies that the weight $\mathcal{W}$ (and the Hessian determinant) are κ-mixed homogeneous of degree 0, and hence bounded on $\mathbb{R}^2$, and so the estimate for $(p_1, p_3) = (2, 1)$ follows trivially by Plancherel.

Finally, let us mention that the most interesting part of the proof of the above theorem is the proof of Fourier restriction for the Normal form (v) from Proposition 1.4.4, which is to be found in Subsection 4.3.5. There we need to estimate the Fourier transform of a certain measure, and for this we perform a natural decomposition of this measure. What is remarkable is that at the critical frequencies only $\mathcal{O}(1)$ decomposition pieces contribute to the size of the Fourier transform. Interestingly, the same thing already happens in the much easier case of Normal form (iv).

In the case of the other weight (1.4.3) (which has no roots away from the origin) we have:

Theorem 1.4.2. *Let ϕ be mixed homogeneous satisfying condition (H1). Let μ be the measure defined as in* (1.4.2) *with $\mathcal{W}(x) = |x|_\kappa^{\mathcal{D}_\mathcal{W}}$. If the exponents $(p_1, p_3) \in (1,2)^2$ and $\mathcal{D}_\mathcal{W} \in \mathbb{R}$ satisfy*

$$\frac{1}{p_1'} + \frac{h_{lin}^{gl}(\phi)}{p_3'} \leqslant \frac{1}{2}, \qquad \frac{1}{p_3'} \leqslant \frac{1}{2h^{gl}(\phi)}, \qquad \mathcal{D}_\mathcal{W} = 2\Big(\frac{|\kappa|}{p_1'} + \frac{\mathcal{D}}{p_3'} - \frac{|\kappa|}{2}\Big),$$

then the Fourier restriction estimate (1.4.1) *holds true.*

We remark that the quantity $\mathcal{D}_\mathcal{W}$ in the above theorem is allowed to be negative.

As a special case of Theorem 1.4.1 we obtain:

Corollary 1.4.3. *Let ϕ be any mixed homogeneous polynomial in $\mathbb{R}^2$ and let μ be the measure defined as in* (1.4.2) *with $\mathcal{W}(x) = |\mathcal{H}_\phi(x)|^{1/4}$. Then the Fourier restriction estimate* (1.4.1) *holds true for $p_1' = p_3' = 4$.*

In the case of the above corollary we note that the Hessian determinant can either vanish identically, or it does not vanish of infinite order anywhere, since it is necessarily a nonzero mixed homogeneous polynomial. But the case when the Hessian determinant vanishes identically is trivial, so we are indeed within the scope of Theorem 1.4.1.

When one considers "isotropically" homogeneous polynomials (i.e., when $\kappa_1 = \kappa_2$) Corollary 1.4.3 recovers the main result of [22]. The strategy of proof in [22] was to first perform certain decompositions of the surface carried measure in order to get appropriate control over the size of $\nabla\phi$ and the Hessian determinant $\mathcal{H}_\phi$, after which one applies an L^4 argument, as the $L^{4/3}(\mathbb{R}^3) \to L^2(\mathrm{d}\mu)$ Fourier restriction estimate is equivalent to the $L^2(\mathrm{d}\mu) \to L^4(\mathbb{R}^3)$ extension estimate.

In contrast, our proofs of Theorem 1.4.1 and Theorem 1.4.2 are based on the following classification of local normal forms:

Proposition 1.4.4. *Let $v \in \mathbb{R}^2 \backslash \{0\}$, let ϕ be as above κ-mixed homogeneous of degree $\mathcal{D}$, and let us assume that it satisfies condition (H1) and that it is degenerate at v (i.e., the Hessian determinant vanishes at v). Then, after a linear transformation of coordinates, the function $\phi_v(x) := \phi(x+v) - \phi(v) - x \cdot \nabla\phi(v)$ and its Hessian determinant $\mathcal{H}_{\phi_v}$ assume precisely one of the following local normal forms at the origin:*

(i) $\phi_v(x) = x_2^k r(x) + \varphi(x)$, $k \geqslant 2$, *φ is flat,*
$\mathcal{H}_{\phi_v}(x) = x_2^{\tilde{k}+2k-2} q(x)$, $\tilde{k} \geqslant 0$, *or $\mathcal{H}_{\phi_v}$ is flat,*
and in case when $\mathcal{H}_{\phi_v}$ is not flat, then φ vanishes identically,

(ii) $\phi_v(x) = x_1^2 r_1(x_1) + x_2^k r_2(x)$, $k \geqslant 3$,
$\mathcal{H}_{\phi_v}(x) = x_2^{k-2} q(x)$,

(iii) $\phi_v(x) = x_1^2 r_1(x) + x_2^k r_2(x)$, $k \geqslant 3$,
$\partial_2^j r_1(0) = c(\phi, v)\, j\, \partial_2^{j-1} r_1(0)$, $j = 1, \ldots, k-1$, *for some constant* $c(\phi, v) \neq 0$,
$\mathcal{H}_{\phi_v}(x) = x_2^{k-2} q(x)$,

(iv) $\phi_v(x) = x_1^2 r_1(x_1) + (x_2 - x_1^2 \psi(x_1))^k r_2(x)$, $k \geqslant 3$,
$\mathcal{H}_{\phi_v}(x) = (x_2 - x_1^2 \psi(x_1))^{k-2} q(x)$,

(v) $\phi_v(x) = x_1^2 r_1(x) + (x_2 - x_1^2 \psi(x_1))^k r_2(x)$, $k \geqslant 3$,
$\partial_2^j r_1(0) = c(\phi, v)\, j\, \partial_2^{j-1} r_1(0)$, $j = 1, \ldots, k-1$, *for some constant* $c(\phi, v) \neq 0$,
$\mathcal{H}_{\phi_v}(x) = (x_2 - x_1^2 \psi(x_1))^{k-2} q(x)$,

(vi) $\phi_v(x) = (x_2 - x_1^2 \psi(x_1))^k r(x)$, $k \geqslant 2$,
$\mathcal{H}_{\phi_v}(x) = (x_2 - x_1^2 \psi(x_1))^{2k-3} q(x)$.

In all the above cases the appearing functions are smooth and do not vanish at the origin, i.e., $r(0), r_1(0), r_2(0), q(0), \psi(0) \neq 0$ (except for the function φ which is flat), and the root of the function $x \mapsto x_2 - x_1^2 \psi(x_1)$ corresponds to the homogeneity curve through v. If condition (H2) is satisfied, then the function φ in case (i) always vanishes identically and the Hessian determinant is never flat. Finally, if $\kappa_1 = \kappa_2$, then only Normal forms (i) and (ii) can appear.

In cases (i) and (ii) one has further subcases (see Subsection 4.2.1) of technical nature, so we left them out in the above proposition. We also note that only in case (vi) the function ϕ_v is not in adapted coordinates (and the adapted coordinates can be achieved only through the nonlinear transformation $(x_1, x_2) \mapsto (x_1, x_2 + x_1^2 \psi(x_1))$), but it is linearly adapted.

The idea to apply Fourier restriction estimates to obtain an a priori estimate for PDEs goes back to R. Strichartz [77]. In our case one can apply the above results to obtain Strichartz estimates for the nonhomogeneous initial problem

$$\begin{cases} (\partial_t - i\phi(D))u(x,t) &= F(x,t), \quad (x,t) \in \mathbb{R}^2 \times (0,\infty), \\ u(x,0) &= G(x), \quad\quad x \in \mathbb{R}^2, \end{cases}$$

where $F \in \mathcal{S}(\mathbb{R}^3)$, $G \in \mathcal{S}(\mathbb{R}^2)$, and $\phi(D)$ denotes the pseudodifferential operator with symbol $\phi(\xi)$. Namely, by an application of the Christ-Kiselev lemma [26] one gets the following result:

Corollary 1.4.5. *Let ϕ, $\mathcal{W}$, and $(p_1, p_3) \in (1,2)^2$ be either as in Theorem 1.4.1 or Theorem 1.4.2, and let us furthermore assume that $\mathcal{D} \in \{0,1\}$. Then for the above non-homogeneous pseudodifferential equation one has the a priori estimate*

$$\|u\|_{L_t^{p_3'}(L_{(x_1,x_2)}^{p_1'})} \leqslant C_1 \|\mathcal{W}^{-1/2}\,\mathcal{F}\,G\|_{L^2(\mathbb{R}^2)} + C_2 \|\,\mathcal{F}^{-1}{}_{(x_1,x_2)}(\mathcal{W}^{-1}\,\mathcal{F}_{(x_1,x_2)}\,F)\|_{L_t^{p_3}(L_{(x_1,x_2)}^{p_1})},$$

where $\mathcal{F}_{(x_1,x_2)}$ is the partial Fourier transformation in the $x = (x_1, x_2)$ direction.

In the case when $\mathcal{W}$ is the function $|\cdot|_\kappa^{\mathcal{D}_\mathcal{W}}$ the norms on the right hand side are a kind of homogeneous anisotropic Sobolev norms [81, Chapter 5] (in particular, note that $\|\mathcal{W}^{-1/2}\,\mathcal{F}\,G\|_{L^2(\mathbb{R}^2)} = \|\,\mathcal{F}^{-1}\,\mathcal{W}^{-1/2}\,\mathcal{F}\,G\|_{L^2(\mathbb{R}^2)}$).

The procedure of how to obtain the corresponding Strichartz estimate from a Fourier restriction estimate is mostly standard and we have added a sketch of the proof of Corollary 1.4.5 in Section 4.5.

1.5 Remarks on notation

For reasons of consistency we use the same notational conventions as in [51]. We use the "variable constant" notation meaning that constants appearing in calculations and in the course of our arguments may have different values on different lines. Furthermore we use the symbols $\sim, \lesssim, \gtrsim, \ll, \gg$ in order to avoid writing down irrelevant constants. If we have two nonnegative quantities A and B, then by $A \ll B$ we mean that there is a sufficiently small positive constant c such that $A \leqslant cB$, by $A \lesssim B$ we mean that there is a (possibly large) positive constant C such that $A \leqslant CB$, and by $A \sim B$ we mean that there are positive constants $C_1 \leqslant C_2$ such that $C_1 A \leqslant B \leqslant C_2 A$. One defines analogously $A \gg B$ and $A \gtrsim B$. Often the constants c and C shall depend on certain parameters p in which case we occasionally write $A \ll_p B$, $A \lesssim_p B$, etc., in order to emphasize this dependence.

A further notational convention adopted from [51] is the use of symbols χ_0 and χ_1 in denoting certain nonnegative smooth compactly supported functions on $\mathbb{R}$. Namely, we require χ_0 to be supported in a neighbourhood of the origin and identically 1 near the origin, and χ_1 to be supported away from the origin and identically 1 on some open neighbourhood of $1 \in \mathbb{R}$. These cutoff functions χ_0 and χ_1 may vary from line to line, and sometimes, when several χ_0 and χ_1 appear within the same formula, they may even designate different functions.

In Chapter 3 the functions r and q (also used with subscripts and tildes) shall be used generically as smooth functions which are nonvanishing at the origin. Occasionally they can also be flat at the origin, in which case we shall state this explicitly.

Chapter 2

Auxiliary results

This chapter contains auxiliary results that we shall often refer to. In the first section we show that one can always ignore the affine terms of ϕ. This shall be particularly important in Chapter 4 when we derive local normal forms for mixed homogeneous surfaces. In Section 2.2 we list results related to oscillatory integrals, such as the van der Corput lemma, and also some results on oscillatory sums from [51] that are useful in conjunction with complex interpolation. In Section 2.3 we state results which we need for handling mixed norms.

2.1 Reduction to the case $\nabla\phi(0) = 0$

In Section 1.2 we mentioned that one can always reduce the mixed normed Fourier restriction problem:

$$\left(\int |\widehat{f}|^2 \rho \mathrm{d}S \right)^{1/2} \leqslant \mathrm{C}(p,\rho,\phi) \|f\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})}, \qquad f \in \mathcal{S}(\mathbb{R}^3), \tag{2.1.1}$$

to the case when $\nabla\phi(0) = 0$, despite rotational invariance not being at one's disposal. Let us justify this. Consider the linear transformation

$$L(x_1,x_2,x_3) := (x_1, x_2, x_3 + \partial_1\phi(0)x_1 + \partial_2\phi(0)x_2)$$

whose inverse and transpose are

$$\begin{aligned} L^{-1}(x_1,x_2,x_3) &= (x_1, x_2, x_3 - \partial_1\phi(0)x_1 - \partial_2\phi(0)x_2), \\ L^t(x_1,x_2,x_3) &= (x_1 + \partial_1\phi(0)x_3, x_2 + \partial_2\phi(0)x_3, x_3). \end{aligned}$$

Plugging in the function $f \circ L^t$ into the expression of the mixed norm Fourier restriction estimate (2.1.1) we obtain

$$\left(\int |\mathcal{F}(f \circ L^t)|^2(\xi,\phi(\xi))\rho(\xi)\sqrt{1+|\nabla\phi(\xi)|^2}\mathrm{d}\xi \right)^{1/2} \leqslant \mathrm{C}_p \, \|f \circ L^t\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})}.$$

Now one just notices that $\|f \circ L^t\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})} = \|f\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})}$, and that

$$\begin{aligned}|\mathcal{F}(f \circ L^t)|^2(\xi, \phi(\xi)) &= |\mathcal{F} f|^2(L^{-1}(\xi, \phi(\xi))) \\ &= |\mathcal{F} f|^2(\xi, \phi(\xi) - \xi \cdot \nabla\phi(0)),\end{aligned}$$

since the determinant of L is 1. Thus the estimate (2.1.1) with the function ϕ is equivalent (up to a slight change in amplitude due to the Jacobian factor $\sqrt{1 + |\nabla\phi(\xi)|^2}$) to the same estimate with the function ϕ replaced by the function $\xi \mapsto \phi(\xi) - \xi \cdot \nabla\phi(0)$, which has gradient 0 at the origin.

2.2 Auxiliary results related to oscillatory sums and integrals

We shall often need the following two one-dimensional oscillatory integral results. The first one is a van der Corput-type estimate used in [51] and originating in the works of van der Corput [29], G. I. Arhipov [2], and J. E. Björk (as noted in [31]).

Lemma 2.2.1. *Let $M \geqslant 2$ be an integer and let $f \in C^M(I)$ be a real-valued function on the interval $I \subset \mathbb{R}$. Let us assume that either*

(i) $|f^{(M)}(s)| \geqslant 1$ for every $s \in I$, or

(ii) f is of polynomial type $M \geqslant 2$, that is, I is compact and there are positive constants c_1, c_2 such that

$$c_1 \leqslant \sum_{j=1}^{M} |f^{(j)}(s)| \leqslant c_2, \quad \textit{for every } s \in I.$$

Then there exists a constant C which depends only on M in case (i), and on M, c_1, c_2, and I in case (ii), such that for every $\lambda \in \mathbb{R}$ we have

$$\Big|\int_I e^{i\lambda f(s)} g(s)\mathrm{d}s\Big| \leqslant C(\|g\|_{L^\infty(I)} + \|g'\|_{L^1(I)})|\lambda|^{-1/M},$$

for any $L^\infty(I)$ function g with an integrable derivative on I. Furthermore, if $G \in L^1(\mathbb{R})$ is a nonnegative function which is majorized by a function $H \in L^1(\mathbb{R})$ such that $\widehat{H} \in L^1(\mathbb{R})$, then for the same constant C as above we have

$$\int_I G(\lambda f(s))\mathrm{d}s \leqslant C(\|H\|_{L^1(\mathbb{R})} + \|\widehat{H}\|_{L^1(\mathbb{R})})|\lambda|^{-1/M}.$$

We note that in the above lemma in case (*ii*) we can use in both expressions $(1 + |\lambda|)^{-1/M}$ instead of $|\lambda|^{-1/M}$ since the constant C depends on I anyway.

We also remark that we can always use $G = |\varphi|$ for a Schwartz function φ since the Fourier transform of $|\varphi|$ is integrable. The proof of this (known) fact is almost straightforward. Namely, the derivative of $|\varphi|$ can have jumps only at the points s where $\varphi(s) = 0$

and $\varphi'(s) \neq 0$. Denote the set of such points N and note that it is a discrete set. In order to estimate the Fourier transform of $|\varphi|$ at ξ, one integrates by parts the expression

$$(\mathcal{F}\,|\varphi|)(\xi) = \int e^{-ix\xi}|\varphi|(x)\mathrm{d}x$$

twice and gets the additional boundary terms which can be estimated by $|\xi|^{-2}\sum_{s\in N}|\varphi'(s)|$. Using the fact that between any two neighbouring points $s_1, s_2 \in N$ there is a point s inbetween such that $\varphi'(s) = 0$ one easily gets $\sum_{s\in N}|\varphi'(s)| \leqslant \int|\varphi''(s)|\mathrm{d}s < +\infty$ and the claim follows.

The second lemma (less general, but with a stronger implication than the one in [51, Section 2.2]) we need gives us an asymptotic of an oscillatory integral of Airy type. We shall also need some variants, but these we shall state and prove along the way when they are needed.

Lemma 2.2.2. *For $\lambda \geqslant 1$ and $u \in \mathbb{R}$, $|u| \lesssim 1$, let us consider the integral*

$$J(\lambda, u, s) := \int_{\mathbb{R}} e^{i\lambda(b(t,s)t^3 - ut)}a(t, s)\mathrm{d}t,$$

where a, b are smooth and real-valued functions on an open neighbourhood of $I \times K$ for I a compact neighbourhood of the origin in $\mathbb{R}$ and K a compact subset of $\mathbb{R}^m$. Let us assume that $b(t, s) \neq 0$ on $I \times K$ and that $|t| \leqslant \varepsilon$ on the support of a. If $\varepsilon > 0$ is chosen sufficiently small and λ sufficiently large, then the following holds true:

(a) If $\lambda^{2/3}|u| \lesssim 1$, then we can write

$$J(\lambda, u, s) = \lambda^{-1/3}g(\lambda^{2/3}u, \lambda^{-1/3}, s),$$

where $g(v, \mu, s)$ is a smooth function of (v, μ, s) on its natural domain.

(b) If $\lambda^{2/3}|u| \gg 1$, then we can write

$$\begin{aligned} J(\lambda, u, s) =& \lambda^{-1/2}|u|^{-1/4}\chi_0(u/\varepsilon) \sum_{\tau\in\{+,-\}} a_\tau(|u|^{1/2}, s; \lambda|u|^{3/2})e^{i\lambda|u|^{3/2}q_\tau(|u|^{1/2},s)} \\ &+ (\lambda|u|)^{-1}E(\lambda|u|^{3/2}, |u|^{1/2}, s), \end{aligned}$$

where $a_\pm$ are smooth functions in $(|u|^{1/2}, s)$ and classical symbols of order 0 in $\lambda|u|^{3/2}$, and where $q_\pm$ are smooth functions such that $|q_\pm| \sim 1$. The function E is a smooth function satisfying

$$|\partial_\mu^\alpha\partial_v^\beta\partial_s^\gamma E(\mu, v, s)| \leqslant C_{N,\alpha,\beta,\gamma}|\mu|^{-N},$$

for all $N, \alpha, \beta, \gamma \in \mathbb{N}_0$.

Proof. For the part (a) we only sketch the proof since it is a straightforward modification of [51, Lemma 2.2., (a)]. In the integral defining J we substitute $t \mapsto \lambda^{-1/3}t$. Then we can write

$$J(\lambda, u, s) = \lambda^{-1/3}\int_{\mathbb{R}} e^{i(b(\lambda^{-1/3}t,s)t^3 - \lambda^{2/3}ut)}a(\lambda^{-1/3}t, s)\chi_0(\lambda^{-1/3}t/\varepsilon)\mathrm{d}t.$$

We added the smooth cutoff function χ_0 localized near 0 in order to emphasize that domain of integration. If we denote

$$\begin{aligned} v &= \lambda^{2/3}u, \\ \mu &= \lambda^{-1/3}, \end{aligned}$$

then the integral can be written as

$$\int_{\mathbb{R}} e^{i(b(\mu t,s)t^3 - vt)} a(\mu t, s)\chi_0(\mu t/\varepsilon)\mathrm{d}t.$$

We split the integral into two parts, depending on whether the integration domain is contained in $|t| \lesssim C$ or $|t| > C$ for some fixed large C, by using a smooth cutoff function. The part where $|t| \lesssim C$ is obviously smooth in all the (bounded) parameters (v, μ, s) and hence it satisfies the conclusion of the lemma. If C is sufficiently large, ε sufficiently small, and $|t| > C$, then

$$\begin{aligned} |\partial_t(b(\mu t, s)t^3 - vt)| &\sim |t|^2, \\ |\partial_t^N \partial^\alpha (b(\mu t, s)t^3 - vt)| &\lesssim_{N,\alpha} |t|^{3+|\alpha|-N}, \end{aligned}$$

where ∂^α is any derivative in the (v, μ, s) variables. Therefore by taking derivatives of the integral in (v, μ, s), factors of polynomial growth in t appear. This can be controlled by using integration by parts a sufficient number of times since the phase derivative is $\sim |t|^2$, and so we get the uniform estimate in this case too.

The part (b) is also a straightforward modification of [51, Lemma 2.2, (b)], and so we sketch the proof. Here we get a stronger result for the function E compared to [51, Lemma 2.2, (b)] since we assume that there are no t^2 terms in the phase. We start by substituting $t \mapsto |u|^{1/2}t$. Then one gets

$$J(\lambda, u, s) = v \int_{\mathbb{R}} e^{i\mu(b(vt,s)t^3 - (\operatorname{sgn} u)t)} a(vt, s)\chi_0(vt/\varepsilon)\mathrm{d}t,$$

where μ denotes $\lambda|u|^{3/2}$ and v denotes $|u|^{1/2}$. If $|u| \gtrsim \varepsilon$ and if ε is sufficiently small, then the integration domain is $|t| \ll 1$, and so we may use integration by parts and get an estimate as is required for the E term in the conclusion.

Let us now assume $|u| \ll \varepsilon$, and so in particular $|v| \ll 1$. The derivative of the phase is

$$\partial_t(b(vt, s)t^3 - (\operatorname{sgn} u)t) = t^2(3b(vt, s) + vt(\partial_1 b)(vt, s)) - \operatorname{sgn}(u).$$

If t is away from the critical points (which only exist if u and b are of the same sign), then we can argue similarly as in the (a) part of the proof by using integration by parts and get an estimate as is required for the E term in the conclusion. If u and b have the same sign, then there are two critical points $|t_\pm(v, s)| \sim 1$. One now applies the stationary phase method at each of the critical points and obtains the form as in the conclusion of the theorem. $\square$

Next, we state results relating the Newton polyhedron and its associated quantities with asymptotics of oscillatory integrals.

Theorem 2.2.3. *Let $\phi : \Omega \to \mathbb{R}$ be a smooth function of finite type defined on an open set $\Omega \subset \mathbb{R}^2$ containing the origin. If Ω is a sufficiently small neighbourhood of the origin and $\eta \in C_c^\infty(\Omega)$, then*

$$\left| \int e^{i(\xi_1 x_1 + \xi_2 x_2 + \xi_3 \phi(x))} \eta(x) \mathrm{d}x \right| \leqslant C_\eta (1 + |\xi|)^{-1/h(\phi)} \left(\log(2 + |\xi|)\right)^{\nu(\phi)},$$

for all $\xi \in \mathbb{R}^3$.

This result was proven in [50] and can be interpreted as a uniform estimate with respect to a linear pertubation of the phase. The case when $h(\phi) < 2$ was considered earlier in [33]. In the case when ϕ is real analytic and there is no pertubation (i.e., $\xi_1 = \xi_2 = 0$) the above result goes back to Varchenko [85]. In the case of a real analytic function ϕ one actually has a uniform estimate with respect to analytic pertubations (this was proved by Karpushkin in [55]).

We also have the following result from [50] which gives us sharpness of Theorem 2.2.3 in the case when $\xi_1 = \xi_2 = 0$.

Theorem 2.2.4. *Let ϕ be as in Theorem 2.2.3 and let us define for $\lambda > 0$ the function*

$$J_\pm(\lambda) := \int e^{\pm i \lambda \phi(x)} \eta(x) \mathrm{d}x$$

for an $\eta \in C_c^\infty(\Omega)$. If the principal face $\pi(\phi^a)$ of $\mathcal{N}(\phi^a)$ is a compact face, and if Ω is a sufficiently small neighbourhood of the origin, then

$$\lim_{\lambda \to +\infty} \frac{\lambda^{1/h(\phi)}}{(\log \lambda)^{\nu(\phi)}} J_\pm(\lambda) = c_\pm \eta(0),$$

where $c_\pm$ are nonzero constants depending on the phase ϕ only.

An analogous result was proved earlier by M. Greenblatt in [39] for real analytic phase functions ϕ. When the principal face is not compact, Theorem 2.2.4 may fail in general (for an example of this see [54]).

Finally, we state three lemmas which we shall often use in conjunction with Stein's complex interpolation theorem. The proofs of the first and the third lemma can be found in [51, Section 2.5], while we only give a brief note on the proof of the second lemma since it is a direct modification of the first one. The proof of all of them are elementary, though the proof of the third one is quite technical.

Lemma 2.2.5. *Let $Q = \prod_{k=1}^n [-R_k, R_k]$ be a compact cuboid in $\mathbb{R}^n$ for some real numbers $R_k > 0$, $k = 1, \ldots, n$, and let $\alpha, \beta^1, \ldots, \beta^n$ be some fixed nonzero real numbers. For a C^1 function H defined on an open neighbourhood of Q, nonzero real numbers $a_1, \ldots, a_n$, and M a positive integer we define*

$$F(t) := \sum_{l=0}^{M} 2^{i\alpha l t} (H\chi_Q)(2^{\beta^1 l} a_1, \ldots, 2^{\beta^n l} a_n)$$

for $t \in \mathbb{R}$. Then there is a constant C which depends only on Q and the numbers α and β^k's, but not on H, a_k's, M, and t, such that

$$|F(t)| \leqslant C \frac{\|H\|_{C^1(Q)}}{|2^{i\alpha t} - 1|}$$

for all $t \in \mathbb{R}$.

We shall often use this lemma in combination with the holomorphic function

$$\gamma(\zeta) := \frac{2^{\alpha(\zeta-1)} - 1}{2^{\alpha(\theta-1)} - 1} \tag{2.2.1}$$

when applying complex interpolation. This function has the property that

$$\left|\gamma(1+it)F(t)\right| \leqslant C_\theta$$

for a positive constant $C_\theta < +\infty$, and $\gamma(\theta) = 1$.

The following lemma is a slight variation of what was written in [51, Remark 2.8].

Lemma 2.2.6. *Let $Q = \prod_{k=1}^n [-R_k, R_k]$ be a compact cuboid in $\mathbb{R}^n$ for some real numbers $R_k > 0$, $k = 1, \ldots, n$, let $\alpha, \beta^1, \ldots, \beta^n$ be some fixed nonzero real numbers, and let $0 < \epsilon < 1$. For a C^1 function H on a neighbourhood of Q, nonzero real numbers $a_1, \ldots, a_n$, and M a positive integer we define*

$$F(t) := \sum_{l=0}^{M} 2^{i\alpha l t} (H\chi_Q)(2^{\beta^1 l} a_1, \ldots, 2^{\beta^n l} a_n)$$

for $t \in \mathbb{R}$. Then there is a constant C which depends only on Q and the numbers α, β^k's, and ϵ, but not on H, a_k's, M, and t, such that

$$|F(t)| \leqslant C \frac{|H(0)| + \sum_{k=1}^n C_k}{|2^{i\alpha t} - 1|}$$

for all $t \in \mathbb{R}$. The constants C_k are given as

$$C_1 := \sup_{y_1 \in R_1} |y_1|^{1-\epsilon} \int_0^1 |(\partial_1 H)(s y_1, 0, \ldots, 0)| \mathrm{d}s,$$
$$C_k := \sup_{y_1, \ldots, y_k} |y_k|^{1-\epsilon} \int_0^1 |(\partial_k H)(y_1, \ldots, y_{k-1}, s y_k, 0, \ldots, 0)| \mathrm{d}s, \quad k > 1,$$

where the supremum goes over the set $\prod_{j=1}^k [-R_j, R_j]$.

The only difference compared to the proof of [51, Lemma 2.7] is that one now writes

$$\begin{aligned} H(y) = H(0) &+ |y_1|^\epsilon \frac{H(y_1, 0, \ldots, 0) - H(0)}{|y_1|^\epsilon} \\ &+ \sum_{k=1}^n |y_k|^\epsilon \frac{H(y_1, \ldots, y_{k-1}, y_k, 0, \ldots, 0) - H(y_1, \ldots, y_{k-1}, 0, 0, \ldots, 0)}{|y_k|^\epsilon}, \end{aligned} \tag{2.2.2}$$

and notes that the fractions are bounded by their respective C_k's.

In the above lemma we could have directly defined C_k's as the Hölder quotients appearing in (2.2.2), but the formulas used in Lemma 2.2.6 turn out to be more practical. One can easily construct an example though where using the Hölder quotients is more appropriate. One example is when one has an oscillatory factor such as in $H(y_1) = y_1^{\epsilon} e^{iy_1^{-1}}$, $0 < y_1 < 1$ (cf. the Riemann singularity as in [75, Chapter VIII, Subsection 1.4.2]). This function is ϵ-Hölder continuous at 0 and satisfies the conclusion of Lemma 2.2.6 in the sense that $|F(t)| \leqslant C/|2^{i\alpha t} - 1|$, but one can show without too much effort that the integral defining C_1 in Lemma 2.2.6 is infinite.

The third lemma is a two parameter version of the first one.

Lemma 2.2.7. *Let $Q = \prod_{k=1}^{n}[-R_k, R_k]$ be a compact cuboid in $\mathbb{R}^n$ for some real numbers $R_k > 0$, $k = 1, \dots, n$, and let $\alpha_1, \alpha_2 \in \mathbb{Q}^{\times}$, and $\beta_1^k, \beta_2^k \in \mathbb{Q}$, $k = 1, \dots, n$, be fixed numbers such that*

$$\alpha_1\beta_2^k - \alpha_2\beta_1^k \neq 0,$$

for all k (i.e., the vector (α_1, α_2) is linearly independent from (β_1^k, β_2^k)). For a C^2 function H defined on an open neighbourhood of Q, nonzero real numbers $a_1, \dots, a_n$, and M_1, M_2 positive integers we define

$$F(t) := \sum_{l_1=0}^{M_1}\sum_{l_2=0}^{M_2} 2^{i(\alpha_1 l_1 + \alpha_2 l_2)t}(H\chi_Q)(2^{(\beta_1^1 l_1 + \beta_2^1 l_2)}a_1, \dots, 2^{(\beta_1^n l_1 + \beta_2^n l_2)}a_n)$$

for $t \in \mathbb{R}$. Then there is a constant C which depends only on Q and the numbers α_1, α_2, β_1^k's, β_2^k's, but not on H, a_k's, M_1, M_2, and t, such that

$$|F(t)| \leqslant C\frac{\|H\|_{C^2(Q)}}{|\mathfrak{q}(t)|}$$

for all $t \in \mathbb{R}$. The function $\mathfrak{q}$ is defined by $\mathfrak{q}(t) := \prod_{j=1}^{N}\tilde{\mathfrak{q}}(jt)\tilde{\mathfrak{q}}(-jt)$, where

$$\tilde{\mathfrak{q}}(t) := (2^{i\alpha_1 t} - 1)(2^{i\alpha_2 t} - 1)\prod_{k=1}^{n}(2^{i(\alpha_1\beta_2^k - \alpha_2\beta_1^k)t} - 1),$$

and N is a positive integer depending on the β_1^k's and β_2^k's.

For future reference we also note the following construction from [51, Remark 2.10] of a complex function γ on the strip $\Sigma := \{\zeta \in \mathbb{C} : 0 \leqslant \operatorname{Re}\zeta \leqslant 1\}$ which shall be used in the context of complex interpolation together with the above two parameter lemma. If we are given $0 < \theta < 1$ and the exponents α_1, α_2, and β_1^k's, β_2^k's as above, we define

$$\gamma(\zeta) := \prod_{j=1}^{N}\frac{\tilde{\gamma}(j(\zeta-1))\tilde{\gamma}(-j(\zeta-1))}{\tilde{\gamma}(j(\theta-1))\tilde{\gamma}(-j(\theta-1))}, \tag{2.2.3}$$

where

$$\tilde{\gamma}(\zeta) := (2^{\alpha_1\zeta} - 1)(2^{\alpha_2\zeta} - 1)\prod_{k=1}^{n}(2^{(\alpha_1\beta_2^k - \alpha_2\beta_1^k)\zeta} - 1).$$

The function γ has the following two key properties. It is an entire analytic function uniformly bounded on the strip Σ, and for the function F as in Lemma 2.2.7 there is a positive constant $C_\theta < +\infty$ such that for all $t \in \mathbb{R}$

$$\Big|\gamma(1+it)F(t)\Big| \leqslant C_\theta.$$

It also has the property that $\gamma(\theta) = 1$.

2.3 Auxiliary results related to mixed L^p-norms

In this subsection R shall denote the Fourier restriction operator $L^p(\mathbb{R}^3) \to L^2(\mathrm{d}\mu)$ for a positive finite Radon measure μ, and all functions and measures will have $\mathbb{R}^3$ as their domain, unless stated otherwise. Recall that we assume $p = (p_1, p_3)$.

We first recall what happens in the simple case when $p = (2,1)$ and μ has the form

$$\langle \mu, f\rangle = \int_\Omega f(x, \phi(x))\, \eta(x)\mathrm{d}x,$$

where ϕ is any measurable function on an open set Ω and $\eta \in C_c^\infty(\Omega)$ is a nonnegative function. In this case the form of the adjoint of R is

$$(R^* f)(x_1, x_2, x_3) = \int_{\mathbb{R}^2} e^{i(x_1\xi_1 + x_2\xi_2 + x_3\phi(\xi))} f(\xi)\eta(\xi)\mathrm{d}\xi,$$

and it is called the extension operator. Using Plancherel for each fixed x_3, we easily get boundedness of $R^* : L^2(\mathrm{d}\mu) \to L^\infty_{x_3}(L^2_{(x_1,x_2)})$. Note that the operator bound depends only on the L^∞ norm of η. In particular we know that $R : L^1_{x_3}(L^2_{(x_1,x_2)}) \to L^2(\mathrm{d}\mu)$ is bounded.

When considering the $L^p - L^2$ Fourier restriction problem for other p's, it is advantageous to reframe the problem using the so called "R^*R" method. The boundedness of the restriction operator $R : L^p \to L^2(\mathrm{d}\mu)$ is equivalent to the boundedness of the operator $T = R^*R$, which can be written as

$$Tf(y) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(y-x)e^{i\xi\cdot x}\, \mathrm{d}\mu(\xi)\, \mathrm{d}x = f * \check{\mu}(y), \qquad f \in \mathcal{S}(\mathbb{R}^3), \tag{2.3.1}$$

in the pair of spaces $L^p \to L^{p'}$, where p' denotes the Young conjugate exponents (p_1', p_3'). Note that the operator T is linear in μ and it even makes sense for a complex μ (unlike the restriction operator R). This enables us to decompose the measure μ into a sum of complex measures, each having an associated operator of the same form as in (2.3.1).

The following few lemmas give us information on the boundedness of convolution operators such as in (2.3.1).

Lemma 2.3.1. *Let us consider the convolution operator $T : f \mapsto f * \widehat{\mu}$ for a tempered Radon measure μ (i.e., a Radon measure which is a tempered distribution).*

(i) If $\widehat{\mu}$ is a measurable function which satisfies

$$|\widehat{\mu}(x_1, x_2, x_3)| \lesssim A(1 + |x_3|)^{-\tilde{\sigma}} \tag{2.3.2}$$

for some $\tilde{\sigma} \in [0, 1)$, then the operator norm of $T : L^p \to L^{p'}$ for $(1/p_1', 1/p_3') = (0, \tilde{\sigma}/2)$ is bounded (up to a multiplicative constant) by A.

(ii) If μ is a bounded function such that $\|\mu\|_{L^\infty} \lesssim B$, then the operator norm of $T : L^2 \to L^2$ is bounded (up to a multiplicative constant) by B.

Proof. One can easily show by integrating (2.3.1) in (x_1, x_2) variables that

$$\|Tf(\cdot, y_3)\|_{L^\infty_{(y_1,y_2)}} \lesssim A \int_{\mathbb{R}} \|f(\cdot, y_3 - x_3)\|_{L^1_{(x_1,x_2)}} (1 + |x_3|)^{-\tilde{\sigma}} \mathrm{d}x_3,$$

and therefore we can now apply the (one-dimensional) Hardy-Littlewood-Sobolev inequality and obtain the claim in the first case. The second case when $p_1 = p_3 = 2$ is a well known classical result for multipliers. □

For a more abstract approach to the above lemma see [38] and [56]. There one also obtains an appropriate result for $\tilde{\sigma} = 1$ when $1/p_1' > 0$, but shall not need this.

A particular useful application of the above lemma is the following.

Lemma 2.3.2. *Let us consider $T : f \mapsto f * \widehat{\mu}$ for a tempered Radon measure μ which is now localized in the frequency space:*

$$\operatorname{supp} \widehat{\mu} \subset \mathbb{R}^2 \times [-\lambda_3, \lambda_3]$$

for a $\lambda_3 \gtrsim 1$. Let us assume that μ and $\widehat{\mu}$ are measurable functions satisfying

$$\begin{aligned} \|\widehat{\mu}\|_{L^\infty} &\lesssim A, \\ \|\mu\|_{L^\infty} &\lesssim B. \end{aligned} \tag{2.3.3}$$

Then T is a bounded operator for $(\frac{1}{p_1'}, \frac{1}{p_3'}) = (0, \frac{\tilde{\sigma}}{2})$ for all $\tilde{\sigma} \in [0, 1)$, with the associated operator norm being at most (up to a multiplicative constant) $A\,\lambda_3^{\tilde{\sigma}}$. The operator norm of $T : L^2 \to L^2$ is bounded (up to a multiplicative constant) by B.

Proof. We only need to obtain the decay estimate (2.3.2). We note that since $\widehat{\mu}$ has x_3 support bounded by λ_3, it follows

$$\begin{aligned} |\widehat{\mu}(x_1, x_2, x_3)| &\lesssim A\,(1 + \lambda_3^{-1}|x_3|)^{-\tilde{\sigma}} \\ &\lesssim A\,\lambda_3^{\tilde{\sigma}}\,(1 + |x_3|)^{-\tilde{\sigma}} \end{aligned}$$

for all $\tilde{\sigma} \in [0, 1)$. □

At the end of this subsection we note the following simple result which tells us that the conclusion of Lemma 2.3.1 is in a sense quite sharp. We remark that the last conclusion in the lemma below is consistent with the condition $\tilde{\sigma} < 1$ in (2.3.2).

Lemma 2.3.3. *Consider the convolution operator $T : f \mapsto f * \widehat{\mu}$ for a tempered Radon measure μ whose Fourier transform $\widehat{\mu}$ is continuous. Let $\varphi : [0, +\infty) \to (0, +\infty)$ be an increasing and unbounded continuous function and assume that at least one of the limits*

$$\lim_{x_3 \to -\infty} \widehat{\mu}(0, 0, x_3) \frac{(1 + |x_3|)^{\tilde{\sigma}}}{\varphi(|x_3|)} \qquad \text{or} \qquad \lim_{x_3 \to +\infty} \widehat{\mu}(0, 0, x_3) \frac{(1 + |x_3|)^{\tilde{\sigma}}}{\varphi(|x_3|)}$$

exists for some $\tilde{\sigma} \in (0, 1)$, with the limiting value being a nonzero number. Then $T : L^p \to L^{p'}$ is not a bounded operator for $(1/p_1', 1/p_3') = (0, \tilde{\sigma}/2)$. The conclusion also holds in the case when φ is the constant function 1, $\tilde{\sigma} = 1$, and if we additionally assume that $\widehat{\mu}$ is an $L^\infty(\mathbb{R}^3)$ function and that both of the above limits exist and are equal, with the limiting value being a nonzero number.

Proof. Let us begin the proof by assuming that the operator

$$T : L_{x_3}^{\frac{2}{2-\tilde{\sigma}}}(L^1_{(x_1,x_2)}) \to L_{x_3}^{2/\tilde{\sigma}}(L^\infty_{(x_1,x_2)})$$

is bounded. Since $\widehat{\mu}$ is continuous, without loss of generality we can assume that

$$\widehat{\mu}(x) \sim |x_3|^{-\tilde{\sigma}}\,\varphi(|x_3|) \tag{2.3.4}$$

for all x in the open set U of the form

$$\{x \in \mathbb{R}^3 : x_3 > K, |(x_1,x_2)| < \epsilon_U(x_3)\},$$

where $K > 0$ and ϵ_U is a continuous and strictly positive function on $\mathbb{R}$.

Now consider the function

$$f(x) = \varepsilon^{-2}\chi_0\Big(\frac{x_1}{\varepsilon}\Big)\chi_0\Big(\frac{x_2}{\varepsilon}\Big)\chi_0\Big(\frac{x_3}{M}\Big),$$

where χ_0 is smooth, identically 1 in the interval $[-1,1]$, and supported within the interval $[-2,2]$. Then

$$\|f\|_{L_{x_3}^{\frac{2}{2-\tilde{\sigma}}}(L^1_{(x_1,x_2)})} \sim M^{1-\frac{\tilde{\sigma}}{2}},$$

and if we assume ε to be sufficiently small and M sufficiently large, one obtains by a simple calculation that

$$Tf(0,0,x_3) \sim \bigg(\chi_0\Big(\frac{\cdot}{M}\Big) * \Big(|\cdot|^{-\tilde{\sigma}}\,\varphi(|\cdot|)\Big)\bigg)(x_3)$$

for all x_3 such that $4M < x_3 < C(M,\varepsilon)$, where $C(M,\varepsilon) \to \infty$ when $\varepsilon \to 0$ and M is fixed. If in addition we know say $x_3 \leqslant 5M < C(M,\varepsilon)$, then

$$Tf(0,0,x_3) \gtrsim M^{1-\tilde{\sigma}}\,\varphi(|M|),$$

and the lower bound on the norm is

$$\begin{aligned}\|Tf\|_{L_{x_3}^{2/\tilde{\sigma}}(L^\infty_{(x_1,x_2)})} &\gtrsim \Big(M^{1-\tilde{\sigma}}\,\varphi(|M|)\Big)\,M^{\tilde{\sigma}/2}\\ &= M^{1-\tilde{\sigma}/2}\,\varphi(|M|).\end{aligned}$$

But now by the boundedness assumption we obtain

$$M^{1-\tilde{\sigma}/2}\,\varphi(|M|) \lesssim M^{1-\tilde{\sigma}/2} \sim \|f\|_{L_{x_3}^{\frac{2}{2-\tilde{\sigma}}}(L^1_{(x_1,x_2)})},$$

i.e., $\varphi(|M|) \lesssim 1$. This is impossible in general since we can take $M \to \infty$.

In the case when the limits are equal, $\tilde{\sigma} = 1$, and φ is the constant function 1, we can take (2.3.4) to be true for $x \in U$ too (after allowing U to contain points with $|x_3| > K$). If we use the same f as above, then for any $x_3 \in [-M/2, M/2]$ we easily obtain from the definition of T that

$$|Tf(0,0,x_3)| \gtrsim \int_K^{M/2} |t|^{-1}\mathrm{d}t - K\|\widehat{\mu}\|_{L^\infty} \gtrsim \ln M$$

for an M sufficiently large and ε sufficiently small. Thus the norm $\|Tf\|_{L^2_{x_3}(L^\infty_{(x_1,x_2)})}$ is bounded below by $M^{1/2}\ln M$, while $\|f\|_{L^2_{x_3}(L^1_{(x_1,x_2)})}$ is of size $M^{1/2}$. This is impossible if T is bounded. ∎

In the case $\tilde{\sigma} = 1$ and when φ is identically equal to a nonzero constant the above proof does not work if the limits have the same absolute value but opposite signs. This is related to the fact that an operator given as a convolution against $x \mapsto x/(1+x^2)$ is bounded $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ since the Fourier transform of $x \mapsto x/(1+x^2)$ is up to a constant $\xi \mapsto e^{-|\xi|} \operatorname{sgn} \xi$.

Chapter 3

Local mixed norm Fourier restriction estimates

In this section we deal with the estimate

$$\left(\int |\widehat{f}|^2 \rho \mathrm{d}S\right)^{1/2} \leqslant \mathrm{C}(p,\rho,\phi)\|f\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})}, \qquad f \in \mathcal{S}(\mathbb{R}^3), \tag{3.0.1}$$

where we remind that S is given as the graph

$$S := \left\{(x_1, x_2, \phi(x_1,x_2)) : x = (x_1,x_2) \in \Omega \subset \mathbb{R}^2\right\}$$

for Ω a small neighbourhood of the origin and ϕ a smooth function of finite type satisfying $\phi(0) = 0$ and $\nabla\phi(0) = 0$, and such that the original coordinate system x is either adapted, or linearly adapted but not adapted. We remind that $\kappa = (\kappa_1, \kappa_2)$ denotes the weight determined by the principal face of $\mathcal{N}(\phi)$ and $m = \kappa_2/\kappa_1 \geqslant 1$. We furthermore remind that when we consider the non-adapted case we assume $h_{\mathrm{lin}}(\phi) < 2$. The function ρ is a $C_c^\infty(S)$ function.

The structure of this chapter is as follows. In Section 3.1 we derive the necessary conditions (by means of Knapp-type examples) for the exponents $p = (p_1, p_3)$ in estimate (3.0.1). See Proposition 3.1.1. In Subsection 3.1.4 we also determine explicitly the Newton polyhedra of ϕ in its original and adapted coordinates in the case when the linear height of ϕ is strictly less than 2. In Section 3.2, Proposition 3.2.2, we deal with the adapted case, i.e., we prove that if ϕ is adapted in its original coordinates, then the estimate (3.0.1) holds for all p's determined by the necessary conditions, except occasionally for a certain endpoint. In the same section (see Proposition 3.2.3) we also reduce the general non-adapted case to considering the part near the principal root jet of ϕ. In Sections 3.3 and 3.4 we handle the case when the linear height of ϕ is strictly less than 2 for a class of functions ϕ which includes all analytic functions (see Theorem 3.3.1 for a precise formulation).

3.1 Necessary conditions

In this section our assumptions on ϕ are as explained at the beginning of this chapter. Our goal is to find a complete set of necessary conditions on $p = (p_1, p_3) \in [1, \infty]^2$ for (3.0.1) to hold true whenever $\rho(0) \neq 0$. We shall reframe the conditions in several ways: an "explicit" form in Subsection 3.1.1, a form as in Theorem 1.3.1 using the Legendre transformation of K in Subsection 3.1.2, and a form when we fix the ratio p_1'/p_3' in Subsection 3.1.3. In Subsection 3.1.4 we discuss the normal forms of ϕ when $h_{\text{lin}}(\phi) < 2$ and determine explicitly the necessary conditions in this case.

3.1.1 The explicit form

Let us first introduce some further notation. First, recall that if ϕ is linearly adapted but not adapted, then the adapted coordinate system is obtained through

$$y_1 = x_1, \quad y_2 = x_2 - \psi(x_1),$$

where ψ is the principal root jet. The function ϕ is in the new coordinates y

$$\phi^a(y_1, y_2) := \phi(y_1, y_2 + \psi(y_1)),$$

i.e., ϕ^a represents the function ϕ in adapted coordinates. We denote the vertices of Newton polyhedron $\mathcal{N}(\phi^a)$ by

$$(A_l, B_l) \in \mathbb{N}_0^2, \quad l = 0, 1, 2, \ldots, n,$$

where $n \geqslant 0$ and we assume that the points are ordered from left to right, i.e., $A_{l-1} < A_l$ for $l = 1, 2, \ldots, n$. Next, we denote the compact edges of $\mathcal{N}(\phi^a)$ by

$$\gamma_l := [(A_{l-1}, B_{l-1}), (A_l, B_l)], \quad l = 1, 2, \ldots, n,$$

and also the unbounded edges by

$$\begin{aligned} \gamma_0 &:= \{(t_1, t_2) \in \mathbb{R}^2 : t_1 = A_0, t_2 \geqslant B_0\}, \\ \gamma_{n+1} &:= \{(t_1, t_2) \in \mathbb{R}^2 : t_1 \geqslant A_n, t_2 = B_n\}, \end{aligned}$$

see Figure 1.2. Let us denote by L_l, $l = 0, \ldots, n+1$, the associated lines on which these edges lie. Each line L_l is given by the equation

$$\kappa_1^l t_1 + \kappa_2^l t_2 = 1,$$

where $(\kappa_1^l, \kappa_2^l) \in [0, \infty)^2$ is its associated weight. We also introduce the quantity

$$a_l := \frac{\kappa_2^l}{\kappa_1^l},$$

which is related to the slope of L_l, namely, its slope is then equal to $-1/a_l$. We obviously have $a_0 = 0$ and $a_{n+1} = \infty$.

Let us denote by $0 < m < \infty$ the leading exponent in the Taylor expansion of ψ. Recall that L_κ is the unique line

$$\kappa_1 t_1 + \kappa_2 t_2 = 1$$

satisfying $\kappa_2 = m\kappa_1$ and which is a supporting line to the Newton polyhedron $\mathcal{N}(\phi^a)$. This line coincides with the line containing the principal face of $\mathcal{N}(\phi)$, as follows from Varchenko's algorithm. Next, let l_0 be such that

$$a_{l_0} > m \geqslant a_{l_0-1}.$$

Note that the point (A_{l_0-1}, B_{l_0-1}) is the right endpoint of the intersection of L_κ and $\mathcal{N}(\phi^a)$. Varchenko's algorithm also shows that $B_{l_0-1} \geqslant A_{l_0-1}$. We denote by l^a the index such that κ^{l^a} is associated to the principal face of $\mathcal{N}(\phi^a)$. If $\pi(\phi^a)$ is a vertex, we take l^a to be associated to the edge to the left of $\pi(\phi^a)$. Note $l^a \geqslant l_0$.

Now we may express the augmented Newton polyhedron $\mathcal{N}^{\text{res}}(\phi^a)$ as the convex hull of the set

$$\mathcal{N}(\phi^a) \cup L_\kappa^+,$$

where L_κ^+ denotes the ray

$$\Big\{(t_1, t_2) \in L_\kappa : t_2 \geqslant B_{l_0-1}\Big\}.$$

Before stating the necessary conditions analogous to [51, Proposition 1.16], let us recall that in the case of the principal face being a vertex, we take κ to determine the line containing the edge of $\mathcal{N}(\phi)$ which has $\pi(\phi)$ as its left endpoint. Furthermore recall that $m = \kappa_2/\kappa_1 \geqslant 1$ and that ϕ is linearly adapted in its original coordinates.

Proposition 3.1.1. *Let ϕ be as above. Let $\rho \geqslant 0$, $\rho \in C_c^\infty(S)$, be a smooth compactly supported function with $\rho(0) \neq 0$, and assume that the estimate* (3.0.1) *holds true. If ϕ is non-adapted, let us consider the nonlinear shear transformation*

$$y_1 := x_1, \quad y_2 := x_2 - \psi(x_1),$$

and let $\phi^a(y) := \phi(y_1, y_2 + \psi(y_1))$ be the function ϕ expressed in the adapted coordinates. Then it necessarily follows that for all weights $(\tilde{\kappa}_1, \tilde{\kappa}_2)$ such that $L_{\tilde{\kappa}}$ is a supporting line to $\mathcal{N}^{res}(\phi^a)$ we have

$$\frac{(1+m)\tilde{\kappa}_1}{p_1'} + \frac{1}{p_3'} \leqslant \frac{\tilde{\kappa}_1 + \tilde{\kappa}_2}{2}. \tag{3.1.1}$$

This is equivalent to

$$\begin{aligned} \frac{(1+m)\kappa_1^l}{p_1'} + \frac{1}{p_3'} &\leqslant \frac{\kappa_1^l + \kappa_2^l}{2}, \qquad l = l_0, \dots, n+1, \\ \frac{(1+m)\kappa_1}{p_1'} + \frac{1}{p_3'} &\leqslant \frac{\kappa_1 + \kappa_2}{2}. \end{aligned} \tag{3.1.2}$$

Furthermore, when ϕ is either adapted or non-adapted we have the conditions

$$\frac{1}{d(\phi)p_1'} + \frac{1}{p_3'} \leqslant \frac{1}{2d(\phi)}, \qquad \frac{1}{p_3'} \leqslant \frac{1}{2h(\phi)}. \tag{3.1.3}$$

In particular when ϕ is non-adapted the first condition in (3.1.3) *then coincides with the one in the second line of* (3.1.2). *Moreover in this case the conditions in* (3.1.2) *for $l > l^a$ are redundant, and if we fix $p_3' = \infty$ (resp. $p_1' = \infty$) then all the conditions reduce to $p_1' \geqslant 2$ (resp. $p_3' \geqslant 2h(\phi)$).*

Proof. We give only a sketch of the proof since it follows the same lines as in [51]. Let us consider any supporting line $L_{\tilde{\kappa}}$ to the augmented Newton polyhedron $\mathcal{N}^{\mathrm{res}}(\phi^a)$ for some weight $(\tilde{\kappa}_1, \tilde{\kappa}_2)$. This particularly implies by the definition of the augmented Newton diagram that $\tilde{\kappa}_2 \geqslant m\tilde{\kappa}_1$.

We first consider the case when $\tilde{\kappa}_1 > 0$, i.e., when the associated line $L_{\tilde{\kappa}}$ is not horizontal. In this case for each sufficiently small $\varepsilon > 0$ we define the region

$$D_\varepsilon^a := \Big\{ y \in \mathbb{R}^2 : |y_1| \leqslant \varepsilon^{\tilde{\kappa}_1}, |y_2| \leqslant \varepsilon^{\tilde{\kappa}_2} \Big\},$$

which in the original coordinate system has the form

$$D_\varepsilon := \Big\{ x \in \mathbb{R}^2 : |x_1| \leqslant \varepsilon^{\tilde{\kappa}_1}, |x_2 - \psi(x_1)| \leqslant \varepsilon^{\tilde{\kappa}_2} \Big\}.$$

Using the $\phi_{\tilde{\kappa}}^a$ part of the Taylor approximation of ϕ^a one easily gets that for each $y \in D_\varepsilon^a$ we have $|\phi^a(y)| \leqslant C\varepsilon$. Returning to the x coordinates we obtain

$$|\phi(x)| \leqslant C\varepsilon, \quad x \in D_\varepsilon.$$

But for $x \in D_\varepsilon$ one has

$$|x_2| \leqslant \varepsilon^{\tilde{\kappa}_2} + |\psi(x_1)| \lesssim \varepsilon^{\tilde{\kappa}_2} + \varepsilon^{m\tilde{\kappa}_1} \lesssim \varepsilon^{m\tilde{\kappa}_1},$$

since $|\psi(x_1)| \lesssim |x_1|^m$ and $\tilde{\kappa}_2 \geqslant m\tilde{\kappa}_1$. Therefore the region D_ε is contained in the set where $|x_1| \leqslant C_1\varepsilon^{\tilde{\kappa}_1}$ and $|x_2| \leqslant C_2\varepsilon^{m\tilde{\kappa}_1}$. Thus we choose a Schwartz function φ_ε which has its Fourier transform of the form

$$\widehat{\varphi_\varepsilon}(x_1, x_2, x_3) = \chi_0\Big(\frac{x_1}{C_1\varepsilon^{\tilde{\kappa}_1}}\Big)\,\chi_0\Big(\frac{x_2}{C_2\varepsilon^{m\tilde{\kappa}_1}}\Big)\,\chi_0\Big(\frac{x_3}{C\varepsilon}\Big),$$

for some smooth compactly supported function χ_0 which is identically 1 on the interval $[-1, 1]$. Then in particular we have $\widehat{\varphi_\varepsilon}(x_1, x_2, \phi(x_1, x_2)) \geqslant 1$ on D_ε.

Now on the one hand, since $\rho(0) \neq 0$, we have

$$\bigg(\int_S |\widehat{\varphi_\varepsilon}|^2 \rho \mathrm{d}S \bigg)^{1/2} \gtrsim |D_\varepsilon|^{1/2} = \varepsilon^{(\tilde{\kappa}_1 + \tilde{\kappa}_2)/2},$$

and on the other

$$\|\varphi_\varepsilon\|_{L_{x_3}^{p_3}(L_{(x_1,x_2)}^{p_1})} \sim \varepsilon^{\frac{(1+m)\tilde{\kappa}_1}{p_1'} + \frac{1}{p_3'}}.$$

Plugging these into (3.0.1) and letting $\varepsilon \to 0$ one obtains (3.1.1) for the non-horizontal edges.

In the horizontal case $\tilde{\kappa}_1 = 0$ one only slightly changes the argument. Namely, one defines for a sufficiently small $\delta > 0$

$$D^a_\varepsilon := \Big\{ y \in \mathbb{R}^2 : |y_1| \leqslant \varepsilon^\delta, |y_2| \leqslant \varepsilon^{\tilde{\kappa}_2} \Big\}.$$

The associated set in the x coordinates D_ε is then contained in the box determined by $|x_1| \leqslant \varepsilon^\delta$ and $|x_2| \leqslant \varepsilon^{m\delta}$. Furthermore, using a Taylor series expansion, one can easily show that for $x \in D_\varepsilon$ we have again $|\phi(x)| \leqslant C\varepsilon$. Now one proceeds as in the non-horizontal case, the only difference is that after taking the limit $\varepsilon \to 0$, one also needs to take the limit $\delta \to 0$.

Let us now briefly explain why (3.1.1) and (3.1.2) are equivalent. We obviously have that (3.1.1) implies (3.1.2). For the reverse implication we note that the $\tilde{\kappa}$'s considered in (3.1.2) are by definition precisely those for which the lines $L_{\tilde{\kappa}}$ contain the edges of the augmented Newton diagram. This means that all the other supporting lines touch the augmented Newton diagram at only one point. Now one just uses the fact that the associated weight $\tilde{\kappa}$ of such a supporting line $L_{\tilde{\kappa}}$ is obtained by a convex combination of weights associated to the edges which intersect at the point through which $L_{\tilde{\kappa}}$ passes. Thus, all the conditions in (3.1.1) can be obtained as convex combinations of conditions in (3.1.2).

The proof of (3.1.3) is similar to the one for (3.1.1). One considers the set D_ε defined by $\{x \in \mathbb{R}^2 : |x_1| \leqslant \varepsilon^{\kappa_1}, |x_2| \leqslant \varepsilon^{\kappa_2}\}$ in the case when the principal face of $\mathcal{N}(\phi)$ is compact. If it is not compact, then one uses $\{x \in \mathbb{R}^2 : |x_1| \leqslant \varepsilon^\delta, |x_2| \leqslant \varepsilon^{\kappa_2}\}$. Using the Taylor approximation of $\phi(x)$ one gets that for $x \in D_\varepsilon$ we have $|\phi(x)| \lesssim \varepsilon$. The first condition in (3.1.3) is then obtained by plugging

$$\widehat{\varphi_\varepsilon}(x_1, x_2, x_3) = \chi_0\Big(\frac{x_1}{\varepsilon^{\kappa_1}}\Big)\,\chi_0\Big(\frac{x_2}{\varepsilon^{\kappa_2}}\Big)\,\chi_0\Big(\frac{x_3}{C\varepsilon}\Big),$$

into the estimate (3.0.1) in the compact case. In the non-compact case we just change ε^{κ_1} to ε^δ.

In the adapted case, when $d(\phi) = h(\phi)$, we also get automatically the second condition from the first one. Finally, as was mentioned at the beginning of this section, if ϕ is non-adapted and if we take l such that κ^l is associated to the principal face of $\mathcal{N}(\phi^a)$, then we have $h(\phi) = 1/(\kappa^l_1 + \kappa^l_2)$. Therefore the associated condition to this l in (3.1.2) implies the second condition in (3.1.3).

Let us now prove the remaining claims. When $p_1' = \infty$, then all the conditions indeed reduce to

$$\frac{1}{p_3'} \leqslant \frac{1}{2h(\phi)}$$

since $\kappa^l_1 + \kappa^l_2$ is minimal precisely for the edge γ_{l^a} which intersects the bisectrix of $\mathcal{N}(\phi^a)$. This is a direct consequence of the fact that the augmented Newton polyhedron is obtained by the intersection of upper half-planes which have L_κ and L_l's with $\kappa^l_2/\kappa^l_1 > m$ (i.e., for $l \geqslant l_0$) as boundaries, and of the fact that the bisectrix intersects L_l at $(1/(\kappa^l_1+\kappa^l_2), 1/(\kappa^l_1+\kappa^l_2))$.

When $p_3' = \infty$, then the condition

$$\frac{1}{p_1'} \leqslant \frac{1}{2}$$

is the strongest one; this is a direct consequence of $\kappa_2^l/\kappa_1^l > m = \kappa_2/\kappa_1$.

We finally prove that one does not need to consider all the conditions in the first row of (3.1.2), but only for $l = l_0, \dots, l^a$ where l^a is such that γ_{l^a} is the principal face of $\mathcal{N}(\phi^a)$. This follows from the following two facts. Namely, we first note that the line in the $(1/p_1', 1/p_3')$-plane given by

$$\frac{\kappa_1^l + \kappa_2^l}{2} = \frac{(1+m)\kappa_1^l}{p_1'} + \frac{1}{p_3'} \tag{3.1.4}$$

intersects the axis $1/p_1' = 0$ at the point which has the $1/p_3'$ coordinate equal to $(\kappa_1^l + \kappa_2^l)/2$, which is greater than $1/(2h(\phi))$ if $l \neq l^a$, by the previous discussion in the case $p_1' = \infty$. And secondly, as κ_1^l decreases when l increases, the slope of the line (3.1.4) in the $(1/p_1', 1/p_3')$-plane increases with l too. Therefore, in the $(1/p_1', 1/p_3')$-plane the lines given by (3.1.4) and corresponding to necessary conditions associated to any l with $l > l^a$ are lying above the line associated to l^a in the area where $1/p_1' \geqslant 0$. □

In the $(1/p_1', 1/p_3')$-plane the necessary conditions from Proposition 3.1.1 determine a polyhedron which we denote by $\mathcal{P}$ (see Figure 1.4). Let us define the lines

$$\begin{aligned} \tilde{L}_l &:= \left\{ \left(\frac{1}{p_1'}, \frac{1}{p_3'} \right) : \frac{(1+m)\kappa_1^l}{p_1'} + \frac{1}{p_3'} = \frac{\kappa_1^l + \kappa_2^l}{2} \right\}, \qquad l = l_0, \dots, n+1, \\ \tilde{L} &:= \left\{ \left(\frac{1}{p_1'}, \frac{1}{p_3'} \right) : \frac{(1+m)\kappa_1}{p_1'} + \frac{1}{p_3'} = \frac{\kappa_1 + \kappa_2}{2} \right\}, \end{aligned}$$

associated to the necessary conditions. Using arguments similar as in the proof of Proposition 3.1.1, or the Legendre transformation from the following Subsection 3.1.2, one can show that the polyhedron $\mathcal{P}$ is of the form

$$\mathcal{P} = O P P_{l_0} P_{l_0+1} \dots P_{l^a-1} P_{l^a} \tilde{P},$$

i.e., the polyhedron with vertices $O, P, P_{l_0}, P_{l_0+1}, \dots, P_{l^a-1}, P_{l^a}, \tilde{P}$, where the point O is the origin and the other points are as follows. The point P is $(1/2, 0)$ and the point $\tilde{P}$ is $(0, 1/(2h(\phi)))$. The point P_{l_0} is the intersection of $\tilde{L}$ and $\tilde{L}_{l_0}$, and all the other points P_l are given as intersections of the lines $\tilde{L}_l$ and $\tilde{L}_{l-1}$ for $l = l_0 + 1, \dots, l^a$. Hence, an easy calculation shows

$$\begin{aligned} P_{l_0} &= \frac{1}{2} \left(\frac{1}{m+1} \left(1 + \frac{\kappa_2^{l_0} - \kappa_2}{\kappa_1^{l_0} - \kappa_1} \right), \kappa_2 - \kappa_1 \frac{\kappa_2^{l_0} - \kappa_2}{\kappa_1^{l_0} - \kappa_1} \right), \\ P_l &= \frac{1}{2} \left(\frac{1}{m+1} \left(1 + \frac{\kappa_2^l - \kappa_2^{l-1}}{\kappa_1^l - \kappa_1^{l-1}} \right), \kappa_2^{l-1} - \kappa_1^{l-1} \frac{\kappa_2^l - \kappa_2^{l-1}}{\kappa_1^l - \kappa_1^{l-1}} \right), \quad l = l_0 + 1, \dots, l^a. \end{aligned}$$

As in the $p_1 = p_3$ case considered in [51], we expect that the conditions from Proposition 3.1.1 are sharp. This will of course follow if we prove that the Fourier restriction estimate is true within the range they determine. In the adapted case, when $d(\phi) = h(\phi)$, the only condition we obtained was

$$\frac{1}{h(\phi)}\frac{1}{p_1'} + \frac{1}{p_3'} \leqslant \frac{1}{2h(\phi)}. \tag{3.1.5}$$

This condition is sharp as will be shown in Section 3.2, though sometimes the endpoint estimate on the $1/p_3'$ axis will not hold.

3.1.2 The form using the Legendre transformation

As already noted, the necessary conditions can be stated as

$$\frac{(1+m)\tilde{\kappa}_1}{p_1'} + \frac{1}{p_3'} \leqslant \frac{\tilde{\kappa}_1 + \tilde{\kappa}_2}{2},$$

for all $(\tilde{\kappa}_1, \tilde{\kappa}_2)$ such that $L_{\tilde{\kappa}}$ is a supporting line to the augmented Newton polyhedron of ϕ^a. This can be rewritten as

$$\frac{1}{p_3'} \leqslant -\frac{1}{2}\bigg(\Big(\frac{2+2m}{p_1'} - 1\Big)\tilde{\kappa}_1 - \tilde{\kappa}_2\bigg).$$

As in Section 1.3 we denote by K the function associating to each $\tilde{\kappa}_1 \in [0, \kappa_1]$ the $\tilde{\kappa}_2$ such that $L_{\tilde{\kappa}}$ is a supporting line to the augmented Newton polyhedron of ϕ^a, i.e., we have $\tilde{\kappa} = (\tilde{\kappa}_1, K_f(\tilde{\kappa}_1))$. The Legendre transformation of K is given by

$$\mathcal{L}(K)[w] := \sup_{u \in [0,\kappa_1]} (wu - K(u)),$$

and thus we have

$$\frac{1}{p_3'} \leqslant -\frac{1}{2}\mathcal{L}(K)\Big[\frac{2+2m}{p_1'} - 1\Big].$$

We have depicted the graph of K in Figure 1.3.

3.1.3 Conditions when the ratio is fixed

If we fix a ratio $r = p_1'/p_3' \in [0, \infty]$, then we are able to introduce a quantity slight more general than the restriction height $h^{\mathrm{res}}(\phi)$ introduced in [51]. We shall not use this quantity in this thesis, but it may prove useful when considering the mixed norm Fourier restriction for functions ϕ with $h_{\mathrm{lin}}(\phi) \geqslant 2$. The cases $r \in \{0, \infty\}$ are not interesting since we shall prove the associated results in Section 3.2 easily, so we assume that $r \in (0, \infty)$ is fixed. In this case the conditions (3.1.2) can be restated as

$$\frac{(1+m)\tilde{\kappa}_1}{rp_3'} + \frac{1}{p_3'} \leqslant \frac{\tilde{\kappa}_1 + \tilde{\kappa}_2}{2},$$

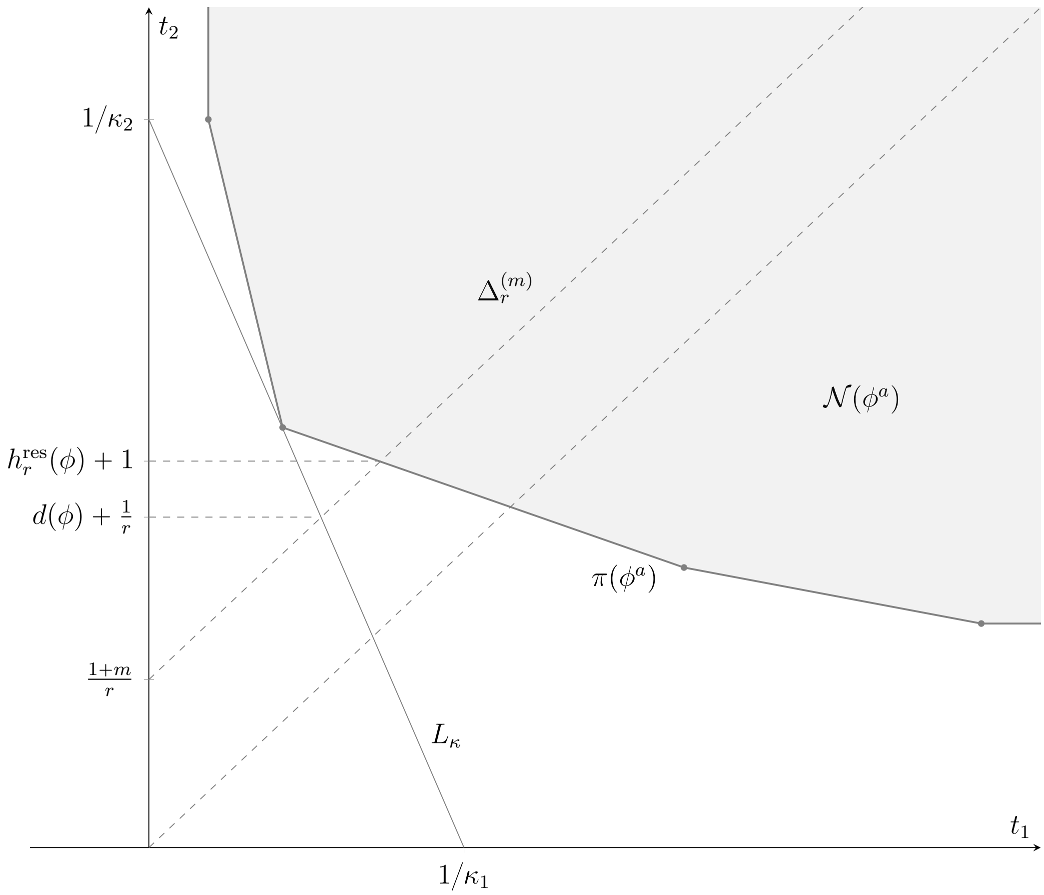

Figure 3.1: The restriction height.

i.e.,

$$p_3' \geqslant 2\,\frac{(1+m)\tilde{\kappa}_1 + r}{r(\tilde{\kappa}_1 + \tilde{\kappa}_2)},$$

where again $\tilde{\kappa}$ is such that $L_{\tilde{\kappa}}$ is a supporting line to the augmented Newton polyhedron $\mathcal{N}^{\mathrm{res}}(\phi^f)$. But now we notice that the number

$$\frac{(1+m)\tilde{\kappa}_1 + r}{r(\tilde{\kappa}_1 + \tilde{\kappa}_2)}$$

is actually the t_2-coordinate of the intersection of the line $L_{\tilde{\kappa}}$ with the parametrized line

$$t \mapsto \left(t - \frac{1+m}{r}, t\right)$$

which we shall denote by $\Delta_r^{(m)}$. This motivates us to define

$$h_r^l := \frac{(1+m)\kappa_1^l + r}{r(\kappa_1^l + \kappa_2^l)} - 1$$

when $\kappa_2^l/\kappa_1^l > m$ (i.e., for $l \geqslant l_0$). Then if we define

$$h_r^{\mathrm{res}}(\phi) := \max\left\{d(\phi) + \frac{1}{r} - 1, h_r^{l_0}, \dots, h_r^{n+1}\right\}, \tag{3.1.6}$$

the conditions (3.1.2) can be restated as the requirement that the inequalities

$$\begin{aligned} p_1' &\geqslant 2r(1 + h_r^{\mathrm{res}}(\phi)), \\ p_3' &\geqslant 2(1 + h_r^{\mathrm{res}}(\phi)), \end{aligned} \tag{3.1.7}$$

must hold necessarily true for all $r \in (0, \infty)$, along with the inequalities $p_1' \geqslant 2$ and $p_3' \geqslant 2h(\phi)$, representing the respective cases $r = 0$ and $r = \infty$.

By definition, the restriction height $h^{\mathrm{res}}(\phi)$ from [51] coincides with $h_r^{\mathrm{res}}(\phi)$ when $r = 1$, and in the same way as in [51] we see from (3.1.6) that $h_r^{\mathrm{res}}(\phi) + 1$ can be read off as the t_2-coordinate of the point where the line $\Delta_r^{(m)}$ intersects the augmented Newton diagram of ϕ^a (see Figure 3.1).

3.1.4 Necessary conditions when $h_{\mathrm{lin}}(\phi) < 2$

In the case when ϕ is non-adapted and the linear height of ϕ is strictly less than 2 it turns out that there are only two necessary conditions from Proposition 3.1.1. Namely, in this case we shall show that $l_0 = l^a$, and therefore the only conditions are

$$\begin{aligned} \frac{(1+m)\kappa_1^{l^a}}{p_1'} + \frac{1}{p_3'} &\leqslant \frac{\kappa_1^{l^a} + \kappa_2^{l^a}}{2}, \\ \frac{(1+m)\kappa_1}{p_1'} + \frac{1}{p_3'} &\leqslant \frac{\kappa_1 + \kappa_2}{2}. \end{aligned}$$

If we replace above the inequality signs with equality signs, we get two linear equations in $(1/p_1', 1/p_3')$. Let $(1/\mathfrak{p}_1', 1/\mathfrak{p}_3')$ be the solution of this system. We shall call $\mathfrak{p} = (\mathfrak{p}_1, \mathfrak{p}_3)$ *the critical exponent.* Then, by interpolation, it is sufficient to prove the Fourier restriction estimate (3.0.1) for the exponent $\mathfrak{p}$ and the endpoint exponents associated to the points lying on the axes, i.e., $(1/2, 0)$ and $(0, 1/(2h(\phi)))$.

In order to obtain what precisely the critical exponent $\mathfrak{p}$ is, we recall [51, Proposition 2.11] which gives us explicit normal forms of ϕ in the case when $h_{\mathrm{lin}}(\phi) < 2$. In the real analytic case these normal forms were derived in [72] by D. Siersma. [51, Proposition 2.11] states that there are two type of singularities, A and D.

In the case of A type singularity the form of the function ϕ is

$$\phi(x_1, x_2) = b(x_1, x_2)(x_2 - \psi(x_1))^2 + b_0(x_1). \tag{3.1.8}$$

Here ψ, b, and b_0 are smooth functions such that $\psi(x_1) = cx_1^m + \mathcal{O}(x_1^{m+1})$ (with $c \neq 0$ and $m \geqslant 2$), $b(0,0) \neq 0$, and $b_0(x_1) = x_1^n\beta(x_1)$ (with either $\beta(0) \neq 0$ and $n \geqslant 2m+1$, or b_0 is flat, i.e., "$n = \infty$"). The function ψ is the principal root jet of ϕ. If b_0 is flat, this is A_∞ type singularity, and otherwise it is A_{n-1} type singularity. In adapted coordinates, the formula (3.1.8) turns into

$$\phi^a(y_1, y_2) = b^a(y_1, y_2)y_2^2 + b_0(y_1), \tag{3.1.9}$$

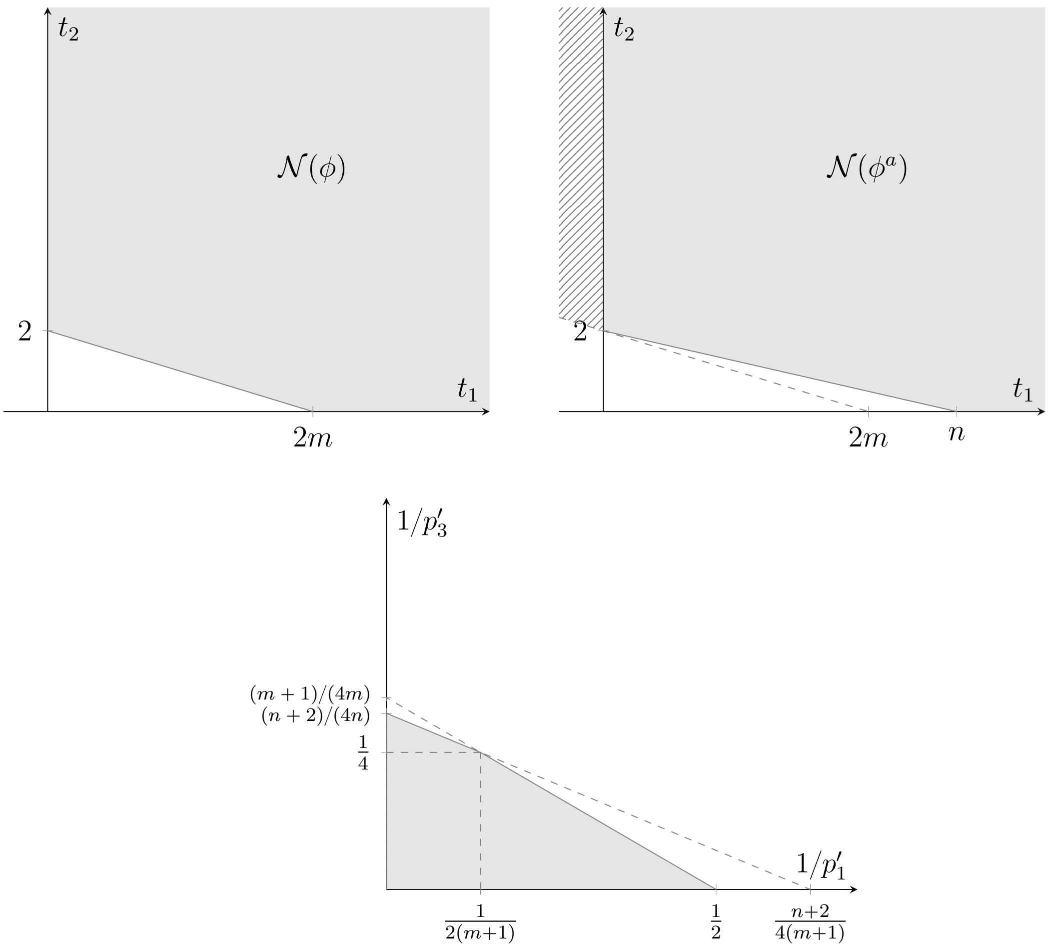

Figure 3.2: The Newton polyhedra associated to A_{n-1} type singularity in the (linearly adapted) original and adapted coordinates respectively, and the associated necessary conditions.

where $b^a(y_1, y_2) = b(y_1, y_2 + \psi(y_1))$, i.e., the function b in (y_1, y_2) coordinates. From the formulas (3.1.8) and (3.1.9) one can now determine the form of the Newton polyhedron of ϕ and ϕ^a (see Figure 3.2). Reading off the Newton polyhedra we have

$$(\kappa_1, \kappa_2) = \Big(\frac{1}{2m}, \frac{1}{2}\Big), \qquad d(\phi) = \frac{2m}{m+1},$$
$$(\kappa_1^{l^a}, \kappa_2^{l^a}) = \Big(\frac{1}{n}, \frac{1}{2}\Big), \qquad h(\phi) = \frac{2n}{n+2},$$

and so the necessary conditions (3.1.2) can be written as

$$\frac{2}{p_1'} + \frac{4m}{m+1}\frac{1}{p_3'} \leqslant 1,$$
$$\frac{4(m+1)}{n+2}\frac{1}{p_1'} + \frac{4n}{n+2}\frac{1}{p_3'} \leqslant 1.$$

Now an easy calculation shows that $(1/\mathfrak{p}_1', 1/\mathfrak{p}_3') = (1/(2m+2), 1/4)$, i.e., we have determined the critical exponent.

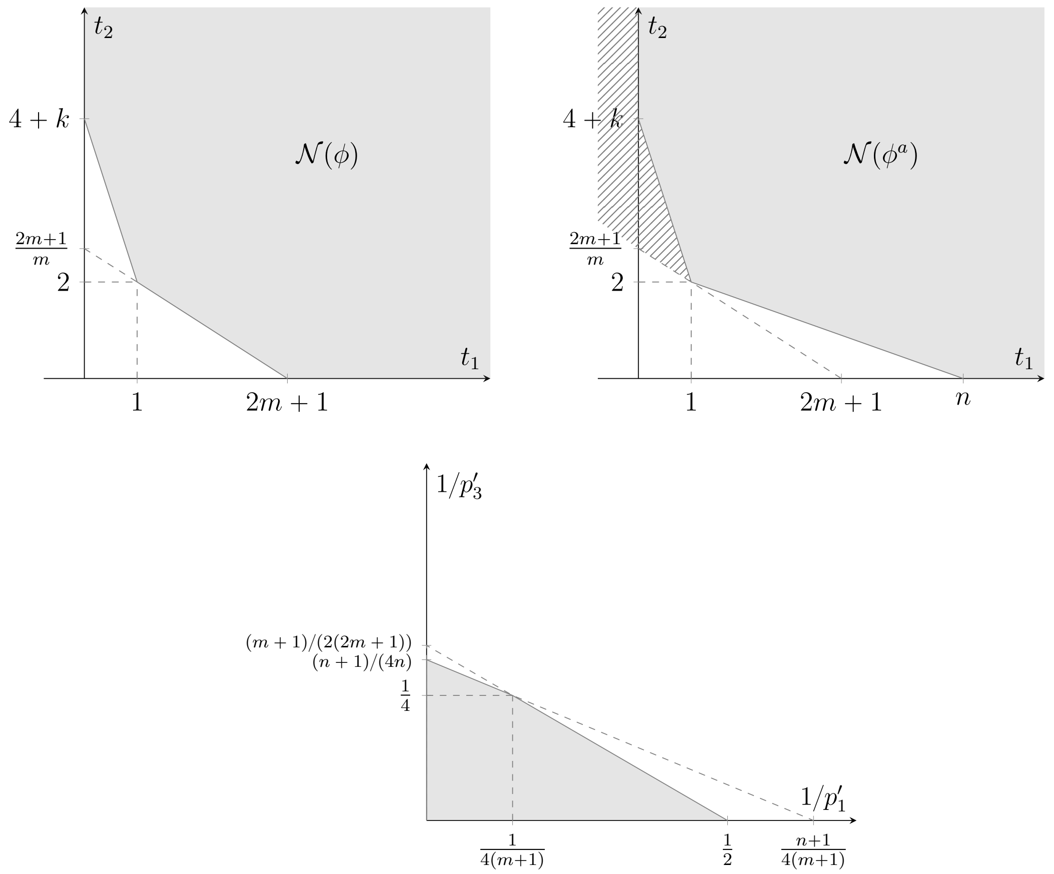

Figure 3.3: The Newton polyhedra associated to D_{n+1} type singularity in the (linearly adapted) original and adapted coordinates respectively, and the associated necessary conditions.

In the case of D type singularity [51, Proposition 2.11] tells us that

$$\begin{aligned} \phi(x_1,x_2) &= (x_1 b_1(x_1,x_2) + x_2^2 b_2(x_2))(x_2-\psi(x_1))^2 + b_0(x_1), \\ \phi^a(y_1,y_2) &= \Big(y_1 b_1^a(y_1,y_2) + (y_2+\psi(y_1))^2 b_2(y_2+\psi(y_1))\Big) y_2^2 + b_0(y_1), \end{aligned} \tag{3.1.10}$$

i.e., the function b from (3.1.8) is now to be written as $b(x_1,x_2) = x_1 b_1(x_1,x_2) + x_2^2 b_2(x_2)$. In this case we have the conditions $b_1(0,0) \neq 0$ and $b_2(x_2) = c_2 x_2^k + \mathcal{O}(x_2^{k+1})$. Again $\psi(x_1) = c x_1^m + \mathcal{O}(x_1^{m+1})$ ($c \neq 0$, $m \geqslant 2$) and $b_0(x_1) = x_1^n \beta(x_1)$, but now either $\beta(0) \neq 0$ and $n \geqslant 2m+2$, or b_0 is flat. If b_0 is flat, this is D_∞ type singularity, and otherwise it is D_{n+1} type singularity. The function b_1^a is the function b_1 in (y_1,y_2) coordinates.

Now one determines the form of the Newton polyhedra (see Figure 3.3) and reads off that

$$\begin{aligned} (\kappa_1,\kappa_2) &= \Big(\frac{1}{2m+1}, \frac{m}{2m+1}\Big), & d(\phi) &= \frac{2m+1}{m+1}, \\ (\kappa_1^{l^a},\kappa_2^{l^a}) &= \Big(\frac{1}{n}, \frac{n-1}{2n}\Big), & h(\phi) &= \frac{2n}{n+1}. \end{aligned}$$

Therefore, the necessary conditions can be written as

$$\begin{aligned} \frac{2}{p_1'} + \frac{2(2m+1)}{m+1}\frac{1}{p_3'} &\leqslant 1, \\ \frac{4(m+1)}{n+1}\frac{1}{p_1'} + \frac{4n}{n+1}\frac{1}{p_3'} &\leqslant 1. \end{aligned}$$

Again, a simple calculation shows that $(1/\mathfrak{p}_1', 1/\mathfrak{p}_3') = (1/(4m+4), 1/4)$.

Note that in the A_∞ and D_∞ cases the necessary conditions form a right-angled trapezium in the $(1/p_1', 1/p_3')$-plane (easily seen by taking $n \to \infty$; one can also do a direct calculation). As the critical exponents in the cases A_{n-1} and D_{n+1} do not depend on n, one is easily convinced that the critical exponents of A_∞ and D_∞ cases are equal to the respective critical exponents of A_{n-1} and D_{n+1}.

3.2 The adapted case and a reduction for the non-adapted case

Here we mimic [51, Chapter 3] and the last section of [50], where the adapted case for $p_1 = p_3$ was considered. In this section we shall be concerned with measures of the form

$$\langle \mu, f \rangle = \int f(x, \phi(x))\eta(x)\mathrm{d}x, \tag{3.2.1}$$

where $\phi(0) = 0$, $\nabla\phi(0) = 0$, and η is a smooth nonnegative function with support contained in a sufficiently small neighbourhood of 0. We assume that ϕ is of finite type on the support of η. The associated Fourier restriction problem is

$$\left(\int |\widehat{f}|^2 \mathrm{d}\mu \right)^{1/2} \leqslant \mathrm{C} \, \|f\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})}, \qquad f \in \mathcal{S}(\mathbb{R}^3), \tag{3.2.2}$$

for any η with support contained in a sufficiently small neighbourhood of 0.

The following proposition will be useful in this section.

Proposition 3.2.1. *Let μ, ϕ, and η be as above. Then the mixed norm Fourier restriction estimate* (3.2.2) *holds true for the point $(1/p_1', 1/p_3') = (1/2, 0)$. Furthermore we have the following two cases.*

(i) If either $h(\phi) = 1$ or $\nu(\phi) = 1$, then the estimate (3.2.2) *holds true for $1/p_1' = 0$ and $1/p_3' < 1/(2h(\phi))$. In this case the estimate for $(1/p_1', 1/p_3') = (0, 1/(2h(\phi)))$ does not hold if $\eta(0) \neq 0$.*

(ii) If $h(\phi) > 1$ and $\nu(\phi) = 0$, then the estimate (3.2.2) *holds true for $(1/p_1', 1/p_3') = (0, 1/(2h(\phi)))$.*

Proof. The claim for $(1/p_1', 1/p_3') = (1/2, 0)$ follows from considerations at the beginning of Section 2.3.

Let us now recall what happens in the non-degenerate case, i.e., when the determinant of the Hessian $\det \mathcal{H}_\phi(0,0) \neq 0$. This is equivalent to $h(\phi) = 1$ and in this case ϕ is adapted in any coordinate system. Here we have the bound (3.2.2) for all of the $(1/p_1', 1/p_3')$ given in the necessary condition (3.1.5), except for the point $(0, 1/2)$, for which it does not hold. This fact is actually true globally, i.e., the Strichartz estimates hold (see [38, 56] and references therein) in the same range, and one can easily convince oneself that the same proof as in say [56] goes through in our local case. For the negative results at the point $(0, 1/2)$ in the case of Strichartz estimates see [59] and [63]. We can also get a negative result at the point $(0, 1/2)$ directly in our case by applying Lemma 2.3.3 for the case $\tilde{\sigma} = 1$ and φ is identically equal to 1. The limits in Lemma 2.3.3 are obtained by a simple application of the two dimensional stationary phase method. Furthermore, since the Hessian does not change its sign when changing the phase $\phi \mapsto -\phi$, the limits in both directions are equal.

The claims for the case when $h(\phi) > 1$ follow easily by applying Theorems 2.2.3 and 2.2.4 to Lemmas 2.3.1 and 2.3.3 respectively. In Lemma 2.3.3 we take φ to be the logarithmic function $x \mapsto \log(2 + x)$. $\square$

3.2.1 The adapted case

The following proposition tells us precisely when the Fourier restriction estimate holds in the adapted case.

Proposition 3.2.2. *Let us assume that μ, ϕ, and η are as explained at the beginning of this section, and let us assume that ϕ is adapted.*

(i) If $h(\phi) = 1$ or $\nu(\phi) = 1$, then the full range Fourier restriction estimate given by the necessary condition (3.1.5) *holds true, except for the point $(1/p_1', 1/p_3') = (0, 1/(2h(\phi)))$ where it is false if $\eta(0) \neq 0$.*

(ii) If $h(\phi) > 1$ and $\nu(\phi) = 0$, then the full range Fourier restriction estimate given by the necessary condition (3.1.5) *holds true, including the point $(1/p_1', 1/p_3') = (0, 1/(2h(\phi)))$.*

Proof. The case when $h(\phi) = 1$ is the classical known case and it was already discussed in the proof of Proposition 3.2.1. The case when $h(\phi) > 1$ and $\nu(\phi) = 0$ follows from Proposition 3.2.1 by interpolation.

Let us now consider the remaining case when $h(\phi) > 1$ and $\nu(\phi) = 1$. Then if we would use Proposition 3.2.1 and interpolation as in the previous case, we would miss all the boundary points determined by the line of the necessary condition (3.1.5)

$$\frac{1}{h(\phi)} \frac{1}{p_1'} + \frac{1}{p_3'} = \frac{1}{2h(\phi)},$$

except the point $(1/2, 0)$ where we know that the estimate always holds. Recall that this is essentially because we have the logarithmic factor in the decay of the Fourier transform of μ. Instead, one can use the strategy from [50, Section 4] to avoid this problem. We only briefly sketch the argument. One decomposes

$$\mu = \sum_{k \geqslant k_0} \mu_k,$$

where μ_k are supported within ellipsoid annuli centered at 0 and closing in to 0. This is done by considering the partition of unity

$$\eta(x) = \sum_{k \geqslant k_0} \eta_k(x) = \sum_{k \geqslant k_0} \eta(x)\chi \circ \delta_{2^k}(x),$$

where χ is an appropriate $C_c^\infty(\mathbb{R}^2)$ function supported away from the origin and

$$\delta_r(x) = (r^{\kappa_1}x_1, r^{\kappa_2}x_2),$$

where $\kappa = (\kappa_1, \kappa_2)$ is the weight associated to the principal face of $\mathcal{N}(\phi)$. Next, one rescales the measures μ_k and obtains measures $\mu_{0,(k)}$ having the form (3.2.1). These new measures have uniformly bounded total variation and Fourier decay estimate with constants uniform in k:

$$\left|\widehat{\mu_{0,(k)}}(\xi)\right| \lesssim (1 + |\xi|)^{-1/h(\phi)}.$$

Note that there is no logarithmic factor anymore. Now we can use Proposition 3.2.1 and interpolation to obtain the mixed norm Fourier restriction estimate within the range (3.1.5) for each $\mu_{0,(k)}$. As in [50, Section 4], one now easily obtains the bound

$$\int |\widehat{f}|^2 \mathrm{d}\mu_k \lesssim 2^{(|\kappa|+2)k} \|f \circ \delta^e_{2^k}\|^2_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})}, \qquad f \in \mathcal{S}(\mathbb{R}^3),$$

where $\delta^e_r(x_1, x_2, x_3) = (r^{\kappa_1}x_1, r^{\kappa_2}x_2, rx_3)$. The scaling in our mixed norm case is

$$\|f \circ \delta^e_{2^k}\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})} = 2^{-k(\frac{\kappa_1+\kappa_2}{p_1} + \frac{1}{p_3})} \|f\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})},$$

and therefore

$$\begin{aligned}
\int |\widehat{f}|^2 \mathrm{d}\mu_k &\lesssim 2^{k(|\kappa|+2) - 2k(\frac{\kappa_1+\kappa_2}{p_1} + \frac{1}{p_3})} \|f\|^2_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})} \\
&= 2^{k(|\kappa|+2-\frac{2|\kappa|}{p_1} - \frac{2}{p_3})} \|f\|^2_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})} \\
&= 2^{2k|\kappa|(-\frac{1}{2} + \frac{1}{p_1'} + \frac{1}{|\kappa|p_3'})} \|f\|^2_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})} \\
&\leqslant \|f\|^2_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})},
\end{aligned}$$

by the necessary condition

$$\frac{1}{p_1'} + \frac{1}{|\kappa|p_3'} \leqslant \frac{1}{2},$$

and the equalities $d(\phi)|\kappa| = h(\phi)|\kappa| = 1$. The rest of the proof is the same as in [50] if we assume $p_1 > 1$, since then one can use the Littlewood-Paley theorem[1] and the Minkowski inequality (which we can apply since $p_1 = p_2 \leqslant 2$ and $p_3 \leqslant 2$) to sum the above inequality in k. The proof of Proposition 3.2.2 is done. □

[1] Here we don't need a mixed norm Littlewood-Paley theorem since the decomposition is only in the tangential direction where $p_1 = p_2$. Note that the ordering of the mixed norm is important, namely that the outer norm is associated to the normal direction.

3.2.2 Reduction to the principal root jet

In this subsection we make some preliminary reductions for the case when ϕ is not adapted. Recall that we may assume that ϕ is linearly adapted and that we denote by ψ the principal root jet of ϕ. Then we can obtain the adapted coordinates y (after possibly interchanging the coordinates x_1 and x_2) through

$$\begin{aligned} y_1 &= x_1, \\ y_2 &= x_2 - \psi(x_1). \end{aligned}$$

Before stating the last proposition of this section (analogous to [51, Proposition 3.1]) let us recall some notation from [51]. We write

$$\psi(x_1) = b_1 x_1^m + \mathcal{O}(x_1^{m+1}),$$

where $b_1 \neq 0$ and $m \geqslant 2$ by linear adaptedness (see [51, Proposition 1.7]). If F is an integrable function on the domain of η, say $\Omega \subseteq \mathbb{R}^2$, then we denote

$$\mu^F := (F \otimes 1)\mu.$$

If χ_0 denotes a $C_c^\infty(\mathbb{R})$ function equal to 1 in a neighbourhood of the origin, we may define

$$\rho_1(x_1, x_2) = \chi_0\Big(\frac{x_2 - b_1 x_1^m}{\varepsilon x_1^m}\Big),$$

where ε is an arbitrarily small parameter. The domain of ρ_1 is a κ-homogeneous subset of Ω which contains the principal root jet $x_2 = \psi(x_1)$ of ϕ when Ω is contained in a sufficiently small neighbourhood of 0.

Proposition 3.2.3. *Assume ϕ is of finite type on Ω, non-adapted, and linearly adapted (i.e., $d(\phi) = h_{lin}(\phi)$). Let $\varepsilon > 0$ be sufficiently small and let $\mu^{1-\rho_1}$ have support contained in a sufficiently small neighbourhood of* 0. *Then the mixed norm Fourier restriction estimate* (3.2.2) *with respect to the measure $\mu^{1-\rho_1}$ holds true for all $(1/p_1', 1/p_3')$ which satisfy*

$$\begin{aligned} \frac{1}{d(\phi)} \frac{1}{p_1'} + \frac{1}{p_3'} &\leqslant \frac{1}{2d(\phi)}, \\ p_1 &> 1, \end{aligned}$$

i.e., within the range determined by the necessary condition associated to the principal face of $\mathcal{N}(\phi)$, except maybe the boundary points of the form $(0, 1/p_3')$. In particular, it also holds true within the narrower range determined by all of the necessary conditions, excluding maybe the boundary points of the form $(0, 1/p_3')$.

We just briefly mention that the proof of the Proposition 3.2.3 is trivial as soon as one uses the results from [51, Chapter 3]. Analogously to the previous subsection, one decomposes the measure $\mu^{1-\rho_1}$ by using the κ dilations associated to the principal face of $\mathcal{N}(\phi)$. The measures ν_k obtained by rescaling are of the form (3.2.1), have uniformly bounded total variation, and have the Fourier transform decay (with constants uniform in k)

$$|\widehat{\nu}_k(\xi)| \lesssim (1 + |\xi|)^{-d(\phi)}.$$

All of this was proven in [51, Chapter 3]. Therefore we have the Fourier restriction estimate for each ν_k for the points $(1/p_1', 1/p_3') = (1/2, 0)$ and $(1/p_1', 1/p_3') = (0, 1/(2d(\phi)))$. Now one uses again interpolation, the Minkowski inequality, and the Littlewood-Paley theorem, to obtain the claim.

Note that the estimates for the boundary points of the form $(0, 1/p_3')$ can be directly solved for the original measure μ through Proposition 3.2.1.

3.3 The case $h_{\text{lin}}(\phi) < 2$

In the remainder of this chapter we shall be concerned with the proof of:

Theorem 3.3.1. *Let $\phi : \Omega \to \mathbb{R}$ be a smooth function of finite type defined on a sufficiently small neighbourhood Ω of the origin, satisfying $\phi(0) = 0$ and $\nabla\phi(0) = 0$. Let us assume that ϕ is linearly adapted, but not adapted, and that $h_{lin}(\phi) < 2$. We additionally assume that the following holds: Whenever the function b_0 appearing in* (3.1.8)*,* (3.1.9)*,* (3.1.10) *is flat (i.e., when ϕ is A_∞ or D_∞ type singularity), then it is necessarily identically equal to* 0*. In this case, for all smooth $\eta \geqslant 0$ with support in a sufficiently small neighbourhood of the origin the Fourier restriction estimate* (3.2.2) *holds for all p given by the necessary conditions determined in Subsection 3.1.4.*

The above condition on the function b_0 is implied by the Condition (R) from [51] (see [51, Remark 2.12. (c)]).

We begin with some preliminaries. As one can see from the Newton diagrams in Subsection 3.1.4, the assumption in our case $h_{\text{lin}}(\phi) < 2$ implies that $h(\phi) \leqslant 2$. Additionally, we see that $h(\phi) = 2$ implies that we either have A_∞ or D_∞ type singularity. As mentioned in Section 1.2, the Varchenko exponent is 0, i.e., $\nu(\phi) = 0$, if $h(\phi) < 2$. When $h(\phi) = 2$ the equality $\nu(\phi) = 0$ also holds true in our case since the principal faces are noncompact. We conclude that if $h_{\text{lin}}(\phi) < 2$, then by Proposition 3.2.1 we have the mixed norm Fourier restriction estimate (3.2.2) for both of the points $(1/p_1', 1/p_3') = (1/2, 0)$ and $(1/p_1', 1/p_3') = (0, 1/(2h(\phi)))$. Therefore, according to Subsection 3.1.4, by interpolation it remains to prove the estimate (3.2.2) for the respective critical exponents given by

$$\begin{aligned} \Big(\frac{1}{\mathfrak{p}_1'}, \frac{1}{\mathfrak{p}_3'}\Big) &= \Big(\frac{1}{2(m+1)}, \frac{1}{4}\Big) && \text{in case of } A \text{ type singularity,} \\ \Big(\frac{1}{\mathfrak{p}_1'}, \frac{1}{\mathfrak{p}_3'}\Big) &= \Big(\frac{1}{4(m+1)}, \frac{1}{4}\Big) && \text{in case of } D \text{ type singularity,} \end{aligned} \tag{3.3.1}$$

where $m \geqslant 2$ is the principal exponent of ψ from Subsection 3.1.4.

Recall that according to Proposition 3.2.3 we may concentrate on the piece of the measure μ located near the principal root jet:

$$\langle \mu^{\rho_1}, f \rangle = \int_{x_1 \geqslant 0} f(x, \phi(x))\, \eta(x)\, \rho_1(x) \mathrm{d}x,$$

where

$$\rho_1(x) = \chi_0\Big(\frac{x_2 - \omega(0)x_1^m}{\varepsilon x_1^m}\Big) \tag{3.3.2}$$

for an arbitrarily small ε and $\omega(0)x_1^m$ the first term in the Taylor expansion of

$$\psi(x_1) = x_1^m \omega(x_1),$$

where ω is a smooth function such that $\omega(0) \neq 0$.

As we use the same decompositions of the measure μ^{ρ_1} as in [51], we shall only briefly outline the decomposition procedure.

3.3.1 Basic estimates

Before we outline the further decompositions and rescalings of μ^{ρ_1}, we first describe here the general strategy for proving the Fourier restriction estimates for the pieces obtained through these decompositions. All of the pieces ν of the measure μ^{ρ_1} will essentially be of the form

$$\langle \nu, f \rangle = \int f \circ \Phi(x)\, a(x) \mathrm{d}x,$$

where Φ is a phase function and $a \geqslant 0$ an amplitude. The amplitude will usually be compactly supported with support away from the origin. Both Φ and a will depend on various decomposition related parameters. We shall need to prove the Fourier restriction estimate with respect to these measures with estimates being uniform in a certain sense with respect to the appearing decomposition parameters.

At this point one uses the "R^*R" method applied to the measure ν. The resulting operator is T_ν which acts by convolution against the Fourier transform of ν. Now one considers the spectral decomposition $(\nu^\lambda)_\lambda$ of the measure ν so that each functions ν^λ is localized in the frequency space at $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, where $\lambda_i \geqslant 1$ are dyadic numbers for $i = 1, 2, 3$. For such functions ν_λ we shall obtain bounds of the form (2.3.3). By Lemma 2.3.2 then we have the bounds on their associated convolution operators T_ν^λ:

$$\begin{aligned} \|T_\nu^\lambda\|_{L^{2/(2-\tilde\sigma)}_{x_3}(L^1_{(x_1,x_2)}) \to L^{2/\tilde\sigma}_{x_3}(L^\infty_{(x_1,x_2)})} &\lesssim A\lambda_3^{\tilde\sigma}, \\ \|T_\nu^\lambda\|_{L^2 \to L^2} &\lesssim B, \end{aligned} \tag{3.3.3}$$

for all $\tilde\sigma \in [0, 1)$. A and B shall again depend on various decomposition related parameters. If we now define

$$(\theta, \tilde\sigma) := \Big(\frac{1}{m+1}, \frac{m-1}{2m}\Big), \qquad \text{in case of } A \text{ type singularity,} \tag{3.3.4}$$

$$(\theta, \tilde\sigma) := \Big(\frac{1}{2(m+1)}, \frac{m}{2m+1}\Big), \qquad \text{in case of } D \text{ type singularity,} \tag{3.3.5}$$

then interpolating (3.3.3) (θ being the interpolation coefficient) we get precisely the estimate for the critical exponent in (3.3.1) with the bound

$$\|T_\nu^\lambda\|_{L^{\mathrm{p}_3}_{x_3}(L^{\mathrm{p}_1}_{(x_1,x_2)}) \to L^{\mathrm{p}_3'}_{x_3}(L^{\mathrm{p}_1'}_{(x_1,x_2)})} \lesssim A^{1-\theta} B^\theta \lambda_3^{\frac{1}{2}-\theta}. \tag{3.3.6}$$

Now it remains to sum over λ.

When $\theta < 1/4$, we shall be able to always sum absolutely. In the cases when $\theta = 1/4$ and particularly $\theta = 1/3$ (note that both appear only in A type singularity with $m = 3$ and $m = 2$ respectively) we shall need the complex interpolation method developed in [51].

3.3.2 First decompositions and rescalings of μ^{ρ_1}

As in Section 3.2, we use the κ dilatations associated to the principal face of $\mathcal{N}(\phi)$, and subsequently a Littlewood-Paley argument. Then it remains to prove the Fourier restriction estimate for the renormalized measures ν_k of the form

$$\langle \nu_k, f \rangle = \int f(x, \phi(x, \delta))\, a(x, \delta)\, \mathrm{d}x,$$

uniformly in k. As was shown in [51, Section 4.1], the function $\phi(x, \delta)$ has the form

$$\phi(x, \delta) := \tilde{b}(x_1, x_2, \delta_1, \delta_2)(x_2 - x_1^m \omega(\delta_1 x_1))^2 + \delta_3 x_1^n \beta(\delta_1 x_1),$$

where

$$\delta = (\delta_1, \delta_2, \delta_3) := (2^{-\kappa_1 k}, 2^{-\kappa_2 k}, 2^{-(n\kappa_1 - 1)k}),$$

and

$$\tilde{b}(x_1, x_2, \delta_1, \delta_2) = \begin{cases} b(\delta_1 x_1, \delta_2 x_2), & \text{when A type singularity,} \\ x_1 b_1(\delta_1 x_1, \delta_2 x_2) + \delta_1^{2m-1} x_2^2 b_2(\delta_2 x_2), & \text{when D type singularity.} \end{cases}$$

Above the functions b, b_1, b_2, β, and the quantity n are as in Subsection 3.1.4. Recall that $m = \kappa_2/\kappa_1 \geqslant 2$ and so $\delta_2 = \delta_1^m$. The amplitude $a(x, \delta) \geqslant 0$ is a smooth function of (x, δ) supported at

$$x_1 \sim 1 \sim |x_2|.$$

Furthermore, due to the ρ_1 cutoff function, which has a κ-homogeneous domain, we may assume $|x_2 - x_1^m \omega(0)| \ll 1$.

Since we can take k arbitrarily large, the parameter δ approaches 0. This implies that on the domain of integration of a we have that $\tilde{b}(x_1, x_2, \delta_1, \delta_2)$ converges as a function of (x_1, x_2) to $b(0, 0)$ (resp. $b_1(0, 0)x_1$) in C^∞ when $k \to \infty$ and ϕ has A type singularity (resp. D type singularity). The amplitude $a(x, \delta)$ converges in C_c^∞ to $a(x, 0)$. We also recall that according to the assumption in Theorem 3.3.1, we may assume that $\delta_3 = 0$ if "$n = \infty$", i.e., if b_0 is flat in the normal form of ϕ.

The next step is to decompose the (compactly) supported amplitude a into finitely many parts, each localized near a point $v = (v_1, v_2)$ for which we may assume that it satisfies $v_2 = v_1^m \omega(0)$ (by compactness and since in (3.3.2) we can take ε arbitrarily small). The newly obtained measures we denote by ν_δ and their new amplitudes by the same symbol $a(x, \delta) \geqslant 0$:

$$\langle \nu_\delta, f \rangle = \int f(x, \phi(x, \delta))\, a(x, \delta)\, \mathrm{d}x,$$

where now the support of $a(\cdot, \delta)$ is contained in the set $|x - v| \ll 1$.

Since we can use Littlewood-Paley decompositions in the mixed norm case (see [62, Theorem 2], and also [8, 35]), we can now decompose the measure ν_δ in the x_3 direction in the same way as in [51, Section 4.1]. This is achieved by using the cutoff functions

$\chi_1(2^{2j}\phi(x,\delta))$ in order to localize near the part where $|\phi(x,\delta)| \sim 2^{-2j}$. Then it remains to prove the mixed norm estimate (3.2.2) for measures $\nu_{\delta,j}$ with bounds uniform in paramteres $j \in \mathbb{N}$ and $\delta = (\delta_1, \delta_2, \delta_3) \in \mathbb{R}^3$, $\delta_i \geqslant 0$, $i = 1, 2, 3$, where the measures $\nu_{\delta,j}$ are defined through

$$\langle \nu_{\delta,j}, f \rangle := \int_{x_1 \geqslant 0} f(x, \phi(x,\delta))\chi_1(2^{2j}\phi(x,\delta))a(x,\delta)\mathrm{d}x, \tag{3.3.7}$$

where j can be taken sufficiently large and δ sufficiently small. The function $2^{2j}\phi(x,\delta)$ can be written as

$$2^{2j}\phi(x,\delta) = 2^{2j}\tilde{b}(x_1, x_2, \delta_1, \delta_2)\Big(x_2 - x_1^m\omega(\delta_1 x_1)\Big)^2 + 2^{2j}\delta_3 x_1^n \beta(\delta_1 x_1).$$

Following [51], we distinguish three cases: $2^{2j}\delta_3 \ll 1$, $2^{2j}\delta_3 \gg 1$, and the most involed $2^{2j}\delta_3 \sim 1$.

3.3.3 The case $2^{2j}\delta_3 \gg 1$

As was done in [51, Subsection 4.1.1], we change coordinates from (x_1, x_2) to $(x_1, 2^{2j}\phi(x,\delta))$ and subsequently perform a rescaling (which we adjust to our mixed norm case). Then one obtains that the mixed norm Fourier restriction for $\nu_{\delta,j}$ is equivalent to the estimate

$$\int |\widehat{f}|^2 \mathrm{d}\tilde{\nu}_{\delta,j} \leqslant \mathrm{C}\sqrt{\delta_3}2^{2j(1-2/\mathfrak{p}_3')}\|f\|^2_{L^{\mathfrak{p}}(\mathbb{R}^3)}, \qquad f \in \mathcal{S}(\mathbb{R}^3),$$

that is, since $\mathfrak{p}_3' = 4$,

$$\int |\widehat{f}|^2 \mathrm{d}\tilde{\nu}_{\delta,j} \leqslant \mathrm{C}\,\delta_3^{\frac{1}{2}}2^j\|f\|^2_{L^{\mathfrak{p}}(\mathbb{R}^3)}, \qquad f \in \mathcal{S}(\mathbb{R}^3), \tag{3.3.8}$$

where $\tilde{\nu}_{\delta,j}$ is the rescaled measure

$$\langle \tilde{\nu}_{\delta,j}, f \rangle := \int f(x_1, \phi(x,\delta,j), x_2)a(x,\delta,j)\chi_1(x_1)\chi_1(x_2)\mathrm{d}x.$$

The function $a(x,\delta,j)$ has in δ and j uniformly bounded C^l norms for an arbitrarily large $l \geqslant 0$, and the phase function is given by

$$\begin{aligned}\phi(x,\delta,j) :=&\tilde{b}_1\Big(x_1, \sqrt{2^{-2j}x_2 + \delta_3 x_1^n\tilde{\beta}(\delta_1 x_1)}, \delta_1, \delta_2\Big)\\ &\times \sqrt{2^{-2j}x_2 + \delta_3 x_1^n\tilde{\beta}(\delta_1 x_1)} + x_1^m\omega(\delta_1 x_1),\end{aligned} \tag{3.3.9}$$

where $x_1 \sim 1$, $x_2 \sim 1$, and without loss of generality we may assume $\tilde{b}_1(x_1, x_2, 0, 0) \sim 1$ and $\tilde{\beta}(0) \sim 1$; for details see [51, Subsection 4.1.1]. There the phase function $\phi(x,\delta,j)$ was obtained by solving the equation

$$2^{2j}\phi(y,\delta) = 2^{2j}\tilde{b}(y_1, y_2, \delta_1, \delta_2)(y_2 - y_1^m\omega(\delta_1 y_1))^2 - 2^{2j}\delta_3 y_1^n\tilde{\beta}(\delta_1 y_1)$$

in y_2 after substituting $x_1 = y_1$ and $x_2 = 2^{2j}\phi(y, \delta)$.

By using the implicit function theorem one can show that when $\delta \to 0$, then we have the following C^∞ convergence in the (x_1, x_2) variables:

$$\begin{cases} \tilde{b}_1(x_1, x_2, \delta_1, \delta_2) \to b(0,0)^{-1/2}, & \text{in case of A type singularity,} \\ \tilde{b}_1(x_1, x_2, \delta_1, \delta_2) \to (b_1(0,0)x_1)^{-1/2}, & \text{in case of D type singularity.} \end{cases} \tag{3.3.10}$$

In both the A and D type singularity cases we see that $\tilde{b}_1$ does not depend on x_2 in an essential way.

Now we proceed to perform a spectral decomposition of $\tilde{\nu}_{\delta,j}$, i.e., for $(\lambda_1, \lambda_2, \lambda_3)$ dyadic numbers with $\lambda_i \geqslant 1$, $i = 1, 2, 3$, we define the spectrally localized measures ν_j^λ through

$$\begin{aligned} \widehat{\nu_j^\lambda}(\xi_1, \xi_2, \xi_3) :=& \chi_1\Big(\frac{\xi_1}{\lambda_1}\Big)\chi_1\Big(\frac{\xi_2}{\lambda_2}\Big)\chi_1\Big(\frac{\xi_3}{\lambda_3}\Big)\widehat{\tilde{\nu}}_{\delta,j}(\xi) \\ =& \chi_1\Big(\frac{\xi_1}{\lambda_1}\Big)\,\chi_1\Big(\frac{\xi_2}{\lambda_2}\Big)\,\chi_1\Big(\frac{\xi_3}{\lambda_3}\Big) \\ & \times \int e^{-i(\xi_2\phi(x,\delta,j)+\xi_3x_2+\xi_1x_1)}\,a(x,\delta,j)\,\chi_1(x_1)\,\chi_1(x_2)\,\mathrm{d}x. \end{aligned} \tag{3.3.11}$$

We slightly abuse notation in the following way. Whenever $\lambda_i = 1$, then the appropriate factor $\chi_1(\frac{\xi_i}{\lambda_i})$ in the above expression should be considered as a localisation to $|\xi_i| \lesssim 1$, instead of $|\xi_i| \sim 1$.

If we define the operators

$$\tilde{T}_{\delta,j}f := f * \widehat{\tilde{\nu}}_{\delta,j}, \qquad T_j^\lambda f := f * \widehat{\nu_j^\lambda},$$

then we formally have

$$\tilde{T}_{\delta,j} = \sum_\lambda T_j^\lambda,$$

and according to (3.3.8) and by applying the "R^*R" technique we need to prove

$$\|\tilde{T}_{\delta,j}\|_{L^{\mathfrak{p}} \to L^{\mathfrak{p}'}} \lesssim \delta_3^{\frac{1}{2}}\, 2^j. \tag{3.3.12}$$

In case when we are able to obtain this estimate by summing absolutely the operator pieces T_j^λ we shall proceed as explained in Subsection 3.3.1. In this case in order to obtain the (3.3.3) estimates we need an L^∞ bound for $\widehat{\nu_j^\lambda}$, which we shall get from the expression (3.3.11), and an L^∞ bound for ν_j^λ, which we shall derive next.

Using the equation (3.3.11) we get by Fourier inversion

$$\begin{aligned} \nu_j^\lambda(x_1, x_2, x_3) =& \lambda_1\lambda_2\lambda_3 \int \check{\chi}_1(\lambda_1(x_1 - y_1))\,\check{\chi}_1(\lambda_2(x_2 - \phi(y, \delta, j))) \\ & \times \check{\chi}_1(\lambda_3(x_3 - y_2))\,a(y, \delta, j)\,\chi_1(y_1)\,\chi_1(y_2)\,\mathrm{d}y. \end{aligned} \tag{3.3.13}$$

Here we immediately obtain that the L^∞ bound on ν_j^λ is up to a multiplicative constant λ_2 by using the first and the third factor within the integral by substituting $\lambda_1 y_1$ and $\lambda_3 y_2$. On the other hand, one can easily verify that

$$\partial_{y_2}\phi(y, \delta, j) \sim \delta_3^{-1/2}\, 2^{-2j} \ll 1,$$

and hence by substituting $z_1 = \lambda_1 y_1, z_2 = \lambda_2 \phi(y,\delta,j)$, and utilising the first two factors within the integral, we obtain

$$\|\nu_j^\lambda\|_{L^\infty} \lesssim \delta_3^{1/2}\, 2^{2j} \lambda_3,$$

and therefore combining these two estimates we get

$$\|\nu_j^\lambda\|_{L^\infty} \lesssim \min\{\lambda_2,\, \delta_3^{1/2}\, 2^{2j} \lambda_3\}. \tag{3.3.14}$$

It remains to estimate the Fourier side; for this we shall need to consider several cases depending on the relation between λ_1, λ_2, and λ_3. Let us mention that as in [51], here we shall have no problems when absolutely summing the "diagonal" pieces where $\lambda_1 \sim \lambda_2 \sim \delta_3^{1/2} 2^{2j} \lambda_3$. However, unlike in [51], a case appears which is not absolutely summable. This will be a recurring theme in this chapter. It will also indicate that we should take care even when estimates are obtained by integration by parts.

Case 1. $\lambda_1 \ll \lambda_2$ or $\lambda_1 \gg \lambda_2$, and $\lambda_3 \gg \lambda_2$. In this case we can use integration by parts in both x_1 and x_2 in (3.3.11) to obtain

$$\|\widehat{\nu_j^\lambda}\|_{L^\infty} \lesssim \Big(\lambda_3 \max\{\lambda_1, \lambda_2\}\Big)^{-N},$$

for any nonnegative integer N. Therefore, after plugging this estimate and the estimate (3.3.14) into (3.3.3) and (3.3.6), we may sum in all three parameters λ_1, λ_2, and λ_3, after which one obtains an admissible estimate for (3.3.12).

Case 2. $\lambda_1 \ll \lambda_2$ or $\lambda_1 \gg \lambda_2$, and $\lambda_3 \lesssim \lambda_2$. Here it is sufficient to use integration by parts in x_1. Therefore, we have

$$\|\widehat{\nu_j^\lambda}\|_{L^\infty} \lesssim \Big(\max\{\lambda_1, \lambda_2\}\Big)^{-N},$$

for any nonnegative integer N. Again, after interpolating summation of operators T_j^λ is possible in all three parameters.

Case 3. $\lambda_1 \sim \lambda_2$ and $\lambda_3 \ll \delta_3^{-1/2}\, 2^{-2j} \lambda_2$. In this case we see that necessarily $\lambda_1 \gtrsim \delta_3^{\frac{1}{2}} 2^{2j}$. Also we note that if we fix say λ_1, then there are only finitely many dyadic numbers λ_2 such that $\lambda_1 \sim \lambda_2$, and therefore we essentially need to sum in only two parameters in this case. By stationary phase (and integration by parts when away from the critical point) in x_1 and integration by parts in x_2 we get

$$\|\widehat{\nu_j^\lambda}\|_{L^\infty} \lesssim \lambda_1^{-\frac{1}{2}}\, (\delta_3^{-\frac{1}{2}}\, 2^{-2j}\, \lambda_1)^{-N}.$$

The better bound in (3.3.14) is $\delta_3^{1/2}\, 2^{2j} \lambda_3$. Therefore (3.3.6) becomes in our case

$$\begin{aligned}
\|T_j^\lambda\|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)}) \to L^{\mathfrak{p}_3'}_{x_3}(L^{\mathfrak{p}_1'}_{(x_1,x_2)})} &\lesssim \lambda_1^{(\theta-1)(N+1/2)}\, (\delta_3^{\frac{1}{2}}\, 2^{2j})^{N(1-\theta)}\, (\delta_3^{\frac{1}{2}}\, 2^{2j})^{\theta}\, \lambda_3^{\theta}\, \lambda_3^{\frac{1}{2}-\theta} \\
&\lesssim \lambda_1^{(\theta-1)(N+1/2)}\, \lambda_3^{\frac{1}{2}}\, (\delta_3^{\frac{1}{2}}\, 2^{2j})^{N-(N-1)\theta},
\end{aligned}$$

and hence by summation in λ_3 and taking $N = 1$ we get

$$\begin{aligned}
\sum_{\lambda_3 \lesssim \delta_3^{-1/2}\, 2^{-2j} \lambda_1} \|T_j^\lambda\|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)}) \to L^{\mathfrak{p}_3'}_{x_3}(L^{\mathfrak{p}_1'}_{(x_1,x_2)})} &\lesssim \lambda_1^{\theta(N+1/2)-N}\, (\delta_3^{\frac{1}{2}}\, 2^{2j})^{N-(N-1)\theta-\frac{1}{2}} \\
&\lesssim \lambda_1^{3\theta/2-1}\, (\delta_3^{\frac{1}{2}}\, 2^{2j})^{\frac{1}{2}} \\
&\lesssim \lambda_1^{-\frac{1}{2}}\, (\delta_3^{\frac{1}{2}}\, 2^{2j})^{\frac{1}{2}}.
\end{aligned}$$

Now we obviously get the desired result by summation over $\lambda_1 \gtrsim \delta_3^{\frac{1}{2}} 2^{2j}$.

Case 4. $\lambda_1 \sim \lambda_2$ **and** $\lambda_3 \sim \delta_3^{-1/2} 2^{-2j} \lambda_2$. Here we essentially sum in only one parameter. Let us first determine the estimate in (3.3.6).

Subcase a). $1 \leqslant \lambda_1 \lesssim \delta_3^{\frac{3}{2}} 2^{4j}$**.** Here we have by stationary phase in x_1

$$\|\widehat{\nu_j^\lambda}\|_{L^\infty} \lesssim \lambda_1^{-1/2}.$$

Therefore by (3.3.6) we obtain

$$\begin{aligned}\|T_j^\lambda\|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)}) \to L^{\mathfrak{p}_3'}_{x_3}(L^{\mathfrak{p}_1'}_{(x_1,x_2)})} &\lesssim \lambda_1^{\frac{1}{2}(\theta-1)} \, \lambda_1^{\theta} \, \lambda_3^{\frac{1}{2}-\theta} \\ &= \lambda_1^{\frac{3}{2}\theta-\frac{1}{2}} \left(\delta_3^{-\frac{1}{2}} 2^{-2j} \lambda_1\right)^{\frac{1}{2}-\theta} \\ &= \delta_3^{\frac{1}{2}\theta-\frac{1}{4}} \, 2^{2j\theta-j} \, \lambda_1^{\frac{\theta}{2}}.\end{aligned}$$

Subcase b). $\lambda_1 \gg \delta_3^{\frac{3}{2}} 2^{4j}$**.** In this case we have by stationary phase in x_1 and subsequently by the van der Corput lemma (Lemma 2.2.1, (i), with $M = 2$) in the second

$$\|\widehat{\nu_j^\lambda}\|_{L^\infty} \lesssim \delta_3^{3/4} 2^{2j} \lambda_1^{-1},$$

and hence

$$\begin{aligned}\|T_j^\lambda\|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)}) \to L^{\mathfrak{p}_3'}_{x_3}(L^{\mathfrak{p}_1'}_{(x_1,x_2)})} &\lesssim \delta_3^{\frac{3}{4}-\frac{3}{4}\theta} \, 2^{2j-2j\theta} \, \lambda_1^{\theta-1} \, \lambda_1^{\theta} \, \delta_3^{\frac{1}{2}\theta-\frac{1}{4}} \, 2^{2j\theta-j} \, \lambda_1^{\frac{1}{2}-\theta} \\ &= \delta_3^{\frac{1}{2}-\frac{1}{4}\theta} \, 2^{j} \, \lambda_1^{\theta-\frac{1}{2}}.\end{aligned}$$

Now we sum in λ_1 using the estimates obtained in calculations in Subcases a) and b):

$$\begin{aligned}\sum_{\lambda_1 \geqslant 1} \|T_j^\lambda\|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)}) \to L^{\mathfrak{p}_3'}_{x_3}(L^{\mathfrak{p}_1'}_{(x_1,x_2)})} &\lesssim \delta_3^{\frac{1}{2}\theta-\frac{1}{4}} \, 2^{2j\theta-j} \, (\delta_3^{\frac{3}{2}} 2^{4j})^{\frac{\theta}{2}} + \delta_3^{\frac{1}{2}-\frac{1}{4}\theta} \, 2^{j} \, (\delta_3^{\frac{3}{2}} 2^{4j})^{\theta-\frac{1}{2}} \\ &= 2 \cdot \delta_3^{\frac{5\theta-1}{4}} \, 2^{4j\theta-j},\end{aligned}$$

and therefore it remains to see whether this is admissible for (3.3.12):

$$\begin{aligned} & \delta_3^{\frac{5\theta-1}{4}} \, 2^{4j\theta-j} \lesssim \delta_3^{\frac{1}{2}} \, 2^{j} \\ \Longleftrightarrow \qquad & \delta_3^{-\frac{3-5\theta}{4}} \lesssim (2^{2j})^{1-2\theta}.\end{aligned}$$

But recall that $2^{2j}\delta_3 \gg 1$, i.e., $\delta_3^{-1} \ll 2^{2j}$, and notice that $0 < \theta \leqslant 1/3$ implies $0 < (3-5\theta)/4 \leqslant 1-2\theta$. Hence, it is indeed admissible and we are done with this case.

Case 5. $\lambda_1 \sim \lambda_2$ **and** $\lambda_3 \gg \delta_3^{-1/2} 2^{-2j} \lambda_2$**.** Here we have by the stationary phase method in x_1 and integration by parts in x_2

$$\|\widehat{\nu_j^\lambda}\|_{L^\infty} \lesssim \lambda_1^{-\frac{1}{2}} \, (\lambda_3)^{-N},$$

and the bound in (3.3.14) is $\lambda_1 \sim \lambda_2$. Interpolating, we obtain (with a different N)

$$\|T_j^\lambda\|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)}) \to L^{\mathfrak{p}_3'}_{x_3}(L^{\mathfrak{p}_1'}_{(x_1,x_2)})} \lesssim \lambda_1^{(3\theta-1)/2} \lambda_3^{-N}. \tag{3.3.15}$$

Now if $\theta < 1/3$, then we can easily sum in both λ_1 and λ_3. Therefore, we assume in the following that $\theta = 1/3$.

Subcase a). $\lambda_1 \gtrsim \delta_3^{\frac{1}{2}} 2^{2j}$**.** Summing here in λ_1 between $\delta_3^{1/2}\, 2^{2j}$ and $\delta_3^{1/2}\, 2^{2j}\, \lambda_3$, both up to a multiplicative constant, we get

$$\begin{aligned}\sum_{\delta_3^{\frac{1}{2}} 2^{2j} \lesssim \lambda_1 \lesssim \delta_3^{\frac{1}{2}} 2^{2j} \lambda_3} \|T_j^\lambda\|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)}) \to L^{\mathfrak{p}_3'}_{x_3}(L^{\mathfrak{p}_1'}_{(x_1,x_2)})} &\lesssim \lambda_3^{-N} \log_2 \Big(\frac{\delta_3^{1/2}\, 2^{2j}\, \lambda_3}{\delta_3^{1/2}\, 2^{2j}} \Big) \\ &\lesssim \lambda_3^{-N+1}.\end{aligned}$$

Now we may sum in λ_3 to get the desired result.

Subcase b). $1 \leqslant \lambda_1 \ll \delta_3^{\frac{1}{2}} 2^{2j}$**.** Note that here we sum λ_3 over all the dyadic numbers greater than or equal to 1. We can also assume that $\lambda_1 \gg \delta_3^{\frac{1}{2}} 2^j$ since summation in λ_1 in (3.3.15) up to $\delta_3^{\frac{1}{2}} 2^j$ gives the bound $\lambda_3^{-N+1/2} \log_2(\delta_3^{\frac{1}{2}} 2^j)$ which we can sum in λ_3 and then estimate by $\delta_3^{\frac{1}{2}} 2^j$. This is admissible for (3.3.12).

In order to obtain the required bound in the remaining range:

$$\delta_3^{\frac{1}{2}} 2^j \ll \lambda_1 \ll \delta_3^{\frac{1}{2}} 2^{2j}, \qquad\qquad 1 \leqslant \lambda_3,$$

we need to use the complex interpolation technique developed in [51]. For simplicity we assume that $\lambda_1 = \lambda_2$ (we can do this without losing much on generality since for a fixed λ_1 there are only finitely many dyadic numbers λ_2 such that $\lambda_1 \sim \lambda_2$).

We need to consider the following function parametrized by the complex number ζ and the dyadic number λ_3:

$$\mu_\zeta^{\lambda_3} = \gamma(\zeta)\, (\delta_3^{-3/2}\, 2^{-3j})^\zeta \sum_{\delta_3^{1/2} 2^j \ll \lambda_1 \ll \delta_3^{1/2} 2^{2j}} (\lambda_1)^{\frac{1-3\zeta}{2}}\, \nu_j^\lambda, \tag{3.3.16}$$

where

$$\gamma(\zeta) = 2^{-3(\zeta-1)/2} - 1.$$

The associated convolution operator (given by convolution against the Fourier transform of the function $\mu_\zeta^{\lambda_3}$) we denote by $T_\zeta^{\lambda_3}$.

At this point let us mention that whenever we use complex interpolation we shall generically denote by μ_ζ the considered measure parametrized by the complex number ζ, sometimes with an additional superscript, as is in the current case. Similarly, the associated operator shall be denoted by T_ζ, up to possible appearing superscripts.

For $\zeta = 1/3$ we see that

$$\delta_3^{1/2}\, 2^j\, \mu_\zeta^{\lambda_3} = \sum_{\delta_3^{1/2} 2^j \ll \lambda_1 \ll \delta_3^{1/2} 2^{2j}} \nu_j^\lambda,$$

which means, by Stein's interpolation theorem, that it is sufficient to prove

$$\begin{aligned} \|T_{it}^{\lambda_3}\|_{L^{2/(2-\tilde{\sigma})}_{x_3}(L^1_{(x_1,x_2)}) \to L^{2/\tilde{\sigma}}_{x_3}(L^\infty_{(x_1,x_2)})} &\lesssim \lambda_3^{-N}, \\ \|T_{1+it}^{\lambda_3}\|_{L^2 \to L^2} &\lesssim 1, \end{aligned} \tag{3.3.17}$$

for some $N > 0$, with constants uniform in $t \in \mathbb{R}$, and where $\tilde{\sigma} = 1/4$ since $m = 2$, i.e., $\theta = 1/3$ (see (3.3.4)).

The first estimate is trivial in (3.3.17). Namely, since $\widehat{\nu_j^\lambda}$ have essentially disjoint supports, it follows from the formula (3.3.16) and the estimate on the Fourier transform of ν_j^λ that

$$\|\widehat{\mu_{it}^{\lambda_3}}\|_{L^\infty} \lesssim \lambda_3^{-N},$$

for any $N \in \mathbb{N}$, the implicit constant depending of course on N. Now one just uses the results from Section 2.3 (and in particular Lemma 2.3.1).

In order to prove the second estimate in (3.3.17) we shall need to use the oscillatory sum result Lemma 2.2.5. It turns out that the term $(\delta_3^{-3/2} 2^{-3j})^\zeta$ in the definition of $\mu_\zeta^{\lambda_3}$ is redundant, and that we can actually prove the stronger estimate

$$\Bigg\| \gamma(1+it) \sum_{\delta_3^{1/2} 2^j \ll \lambda_1 \ll \delta_3^{1/2} 2^{2j}} (\lambda_1)^{-1-\frac{3}{2}it} \, \nu_j^\lambda \Bigg\|_{L^\infty} \lesssim 1,$$

that is

$$\Bigg\| \sum_{\delta_3^{1/2} 2^j \ll \lambda_1 \ll \delta_3^{1/2} 2^{2j}} (\lambda_1)^{-1-\frac{3}{2}it} \, \nu_j^\lambda \Bigg\|_{L^\infty} \lesssim \frac{1}{\left|2^{-\frac{3}{2}it} - 1\right|}, \tag{3.3.18}$$

uniformly in t.

We start by substituting $\lambda_1 y_1 \mapsto y_1$ and $\lambda_3 y_2 \mapsto y_2$ in the expression (3.3.13) and plugging the obtained expression into the sum on the left hand side of (3.3.18):

$$\begin{aligned} \sum_{\delta_3^{1/2} 2^j \ll \lambda_1 \ll \delta_3^{1/2} 2^{2j}} (\lambda_1)^{-\frac{3}{2}it} \iint & \check{\chi}_1(\lambda_1 x_1 - y_1) \, \check{\chi}_1(\lambda_1 x_2 - \lambda_1 \phi(y_1/\lambda_1, y_2/\lambda_3, \delta, j)) \\ & \times \check{\chi}_1(\lambda_3 x_3 - y_2) \, a(y_1/\lambda_1, y_2/\lambda_3, \delta, j) \\ & \times \chi_1(y_1/\lambda_1) \, \chi_1(y_2/\lambda_3) \, \mathrm{d}y_1 \mathrm{d}y_2. \end{aligned}$$

Recall that here $y_1 \sim \lambda_1$ and $y_2 \sim \lambda_3$ are both positive, and that $|\phi(\lambda_1^{-1} y_1, \lambda_3^{-1} y_2, \delta, j)| \sim 1$. Therefore we can assume $|(x_1, x_2)| \leqslant C$ for some large constant C, since otherwise we can use the first two factors within the integral to gain a factor λ_1^{-N}. As the dominant term in ϕ is in the y_1 variable and as λ_3 is fixed, we shall only concentrate on the y_1 integration and consider $y_2/\lambda_3 \sim 1$ as a bounded parameter. Therefore the inner y_1 integration, after substituting $\lambda_1 x_1 - y_1 \mapsto y_1$, becomes

$$\begin{aligned} \sum_{\delta_3^{1/2} 2^j \ll \lambda_1 \ll \delta_3^{1/2} 2^{2j}} (\lambda_1)^{-\frac{3}{2}it} \int & \check{\chi}_1(y_1) \, \check{\chi}_1(\lambda_1 x_2 - \lambda_1 \phi(x_1 - \lambda_1^{-1} y_1, \lambda_3^{-1} y_2, \delta, j)) \\ & \times a(x_1 - \lambda_1^{-1} y_1, \lambda_3^{-1} y_2, \delta, j) \, \chi_1(x_1 - \lambda_1^{-1} y_1) \mathrm{d}y_1, \end{aligned}$$

where now $x_1 - \lambda_1^{-1} y_1 \sim 1$, and therefore $|y_1| \lesssim \lambda_1$.

Next, we can restrict ourselves, by using a smooth cutoff function, to the discussion of the integration domain where $|y_1| \ll \lambda_1^{\varepsilon}$ for some small ε, since in the other part by using the first factor in the integral we could gain a factor of $\lambda_1^{-N\varepsilon}$. Since $\lambda_1 \gg \delta_3^{1/2} 2^j \gg 1$ can be taken arbitrarily large, and hence $\lambda_1^{-1} y_1$ arbitrarily small, the relation $x_1 - \lambda_1^{-1} y_1 \sim 1$ implies $x_1 \sim 1$. Therefore by applying a Taylor expansion to the function $\phi(x_1 - \lambda_1^{-1} y_1, \lambda_3^{-1} y_2, \delta, j)$ in the first variable, we obtain

$$\sum_{\delta_3^{1/2} 2^j \ll \lambda_1 \ll \delta_3^{1/2} 2^{2j}} (\lambda_1)^{-\frac{3}{2}it} \int \check{\chi}_1(y_1) \check{\chi}_1(\lambda_1 Q(x_1, x_2, \lambda_3^{-1} y_2, \delta, j) + y_1\, r(\lambda_1^{-1} y_1, x_1, \lambda_3^{-1} y_2, \delta, j))$$
$$\times\, a(x_1 - \lambda_1^{-1} y_1, \lambda_3^{-1} y_2, \delta, j)\, \chi_1(x_1 - \lambda_1^{-1} y_1)\, \chi_0(\lambda_1^{-\varepsilon}\, y_1) \mathrm{d}y_1,$$

where $|\partial_1^N r| \sim 1$ for any $N \geqslant 0$, and $Q(x_1, x_2, \lambda_3^{-1} y_2, \delta, j) = x_2 - \phi(x_1, \lambda_3^{-1} y_2, \delta, j)$.

Now we note that the first two factors in the integral are essentially a convolution, and therefore, by using this two factors, one easily obtains that the bound on the integral is $|\lambda_1 Q|^{-N}$. If $|\lambda_1 Q| \gg 1$, $|\lambda_1 Q|^{-N}$ is a geometric series summable in λ_1, and if $|\lambda_1 Q| \lesssim 1$, then we are actually within the scope of Lemma 2.2.5. Namely, we define the function H as

$$\begin{aligned} H(z_1, z_2, z_3; \lambda_3^{-1} y_2, x_1, x_2, \delta, 2^{-j}) := \int & \check{\chi}_1(y_1)\, \check{\chi}_1(z_1 + y_1\, r(z_2^{1/\varepsilon} y_1, x_1, \lambda_3^{-1} y_2, \delta, j)) \\ & \times a(x_1 - z_2^{1/\varepsilon} y_1, \lambda_3^{-1} y_2, \delta, j)\, \chi_1(x_1 - z_2^{1/\varepsilon} y_1) \\ & \times \chi_0(z_2\, y_1) \mathrm{d}y_1. \end{aligned}$$

Note that H does not actually depend on z_3, but we need to use it in order to implement the lower bound on λ_1 in the summation (this is realised through the characteristic function χ_Q in the definition of $F(t)$ in Lemma 2.2.5). Tracing back, we note that all the dependencies in j are actually dependencies in 2^{-j}. All the parameters $(\lambda_3^{-1} y_2, x_1, x_2, \delta, 2^{-j})$ are now restrained to a bounded set and the C^1 norm of H in (z_1, z_2, z_3) is bounded uniformly in all the (bounded) parameters if (z_1, z_2, z_3) are contained in a bounded set. Therefore by taking

$$(z_1, z_2, z_3) = (\lambda_1 Q(x_1, x_2, \lambda_3^{-1} y_2, \delta, j), \lambda_1^{-\varepsilon}, \delta_3^{1/2} 2^j \lambda_1^{-1})$$

and applying Lemma 2.2.5 with $\alpha = -3/2$, $\lambda_1 = 2^l$, $M = c\delta_3^{1/2} 2^{2j}$ for a small $c > 0$ determined by the implicit constant in the summation condition $\lambda_1 \ll \delta_3^{1/2} 2^{2j}$, and with

$$\begin{aligned} (\beta^1, \beta^2, \beta^3) &= (1, -\varepsilon, -1), \\ (a_1, a_2, a_3) &= (Q(x_1, x_2, \lambda_3^{-1} y_2, \delta, j), 1, \delta_3^{1/2} 2^j), \end{aligned}$$

we obtain the bound (3.3.18). Note that the lower bound on λ_1 in the summation in (3.3.18) is realised by taking $|z_3| \ll 1$. We are done with the case $2^{2j} \delta_3 \gg 1$.

3.3.4 The setting when $2^{2j} \delta_3 \lesssim 1$

As explained in Section [51, Subsection 4.2], in this case we use the change of coordinates $(x_1, x_2) \mapsto (x_1, 2^{-j} x_2 + x_1^m \omega(\delta_1 x_1))$ in the expression (3.3.7) for $\nu_{\delta,j}$. After renormalising the measure $\nu_{\delta,j}$ we obtain that the mixed norm Fourier restriction estimate for $\nu_{\delta,j}$ is equivalent to

$$\int |\widehat{f}|^2 \mathrm{d}\tilde{\nu}_{\delta,j} \leqslant \mathrm{C}\, 2^{j(1-4/p_3')} \|f\|^2_{L^{\mathfrak{p}}(\mathbb{R}^3)}, \qquad f \in \mathcal{S}(\mathbb{R}^3),$$

that is, since $\mathfrak{p}_3' = 4$,

$$\int |\widehat{f}|^2 \mathrm{d}\tilde{\nu}_{\delta,j} \leqslant \mathrm{C}\, \|f\|^2_{L^{\mathfrak{p}}(\mathbb{R}^3)}, \qquad f \in \mathcal{S}(\mathbb{R}^3),$$

where $\tilde{\nu}_{\delta,j}$ is the rescaled measure

$$\langle \tilde{\nu}_{\delta,j}, f \rangle := \int f(x_1, 2^{-j}x_2 + x_1^m \omega(\delta_1 x_1), \phi^a(x,\delta,j)) a(x,\delta,j) \mathrm{d}x.$$

As we see, the Fourier restriction inequality is invariant in the mixed norm case with respect to the scaling we applied. This was interestingly not the case when $p_1 = p_2 = p_3$.

The function $a(x,\delta,j)$ has the form

$$a(x,\delta,j) := \chi_1(\phi^a(x,\delta,j)) a(x_1, 2^{-j}x_2 + x_1^m \omega(\delta_1 x_1), \delta)$$

and the phase function is given by

$$\phi^a(x,\delta,j) := \tilde{b}(x_1, 2^{-j}x_2 + x_1^m \omega(\delta_1 x_1), \delta_1, \delta_2) x_2^2 + 2^{2j} \delta_3 x_1^n \beta(\delta_1 x_1), \tag{3.3.19}$$

where $|\tilde{b}(x_1, x_2, 0, 0)| \sim 1$ and $|\beta(0)| \sim 1$.

Also, we recall that when $\delta \to 0$, then $\tilde{b}(x_1, x_2, \delta_1, \delta_2)$ converges in C^∞ to a nonzero constant if ϕ has A type singularity, and that it converges up to a multiplicative constant to x_1 if ϕ has D type singularity. We shall assume without loss of generality that $\tilde{b}(x_1, x_2, \delta_1, \delta_2) > 0$ since one can just reflect the third coordinate of f in the expression for the measure $\tilde{\nu}_{\delta,j}$.

Support assumptions on $a(\cdot,\delta)$ from Subsection 3.3.2 (namely, that the support is contained in a small neighbourhood of the point $(v_1, v_1^m \omega(0))$ for some $v_1 > 0$) imply that $a(\cdot,\delta,j)$ is supported in a set where $x_1 \sim 1$ and $|x_2| \lesssim 1$.

We again perform a spectral decomposition of $\tilde{\nu}_{\delta,j}$, i.e., for $(\lambda_1, \lambda_2, \lambda_3)$ dyadic numbers with $\lambda_i \geqslant 1$, $i = 1,2,3$, we consider localized measures ν_j^λ defined through

$$\begin{aligned} \widehat{\nu_j^\lambda}(\xi) = & \chi_1\Big(\frac{\xi_1}{\lambda_1}\Big) \chi_1\Big(\frac{\xi_2}{\lambda_2}\Big) \chi_1\Big(\frac{\xi_3}{\lambda_3}\Big) \\ & \times \int e^{-i\Phi(x,\delta,j,\xi)}\, a(x,\delta,j)\, \chi_1(x_1)\, \chi_1(x_2)\, \mathrm{d}x, \end{aligned} \tag{3.3.20}$$

with the complete phase function Φ being

$$\Phi(x,\delta,j,\xi) := \xi_3 \phi^a(x,\delta,j) + 2^{-j}\xi_2 x_2 + \xi_2 x_1^m \omega(\delta_1 x_1) + \xi_1 x_1.$$

We also introduce the operators $\tilde{T}_{\delta,j} f := f * \widehat{\tilde{\nu}}_{\delta,j}$ and $T_j^\lambda f := f * \widehat{\nu_j^\lambda}$. Then we need to prove:

$$\|\tilde{T}_{\delta,j}\|_{L^{\mathfrak{p}} \to L^{\mathfrak{p}'}} \lesssim 1. \tag{3.3.21}$$

In most of the cases this will be done in a similar manner as in the previous subsection. In the case when $2^{2j}\delta_3 \sim 1$, $\theta = 1/3$, and $\lambda_1 \sim \lambda_2 \sim \lambda_3$, with which we shall deal in the next Section, we shall need to perform a finer analysis.

3.3.5 The case $2^{2j}\delta_3 \ll 1$

Here we have the stronger bounds $x_1 \sim 1$ and $|x_2| \sim 1$ since $\phi^a(x,\delta,j) \sim 1$ by (3.3.19) and the assumption $2^{2j}\delta_3 \ll 1$. We also have $|\partial_{x_2}\phi^a(x,\delta,j)| \sim 1$ since $\phi^a(x,\delta,j)$ is a small pertubation of $b(0,0)x_2^2$ in case of A type singularity, and a small pertubation of $b_1(0,0)x_1x_2^2$ in case of D type singularity.

Taking the inverse transform of (3.3.20) we get

$$\begin{aligned} \nu_j^\lambda(x) =& \lambda_1\lambda_2\lambda_3 \int \check{\chi}_1(\lambda_1(x_1-y_1))\, \check{\chi}_1(\lambda_2(x_2-2^{-j}y_2-y_1^m\omega(\delta_1 y_1))) \\ & \times \check{\chi}_1(\lambda_3(x_3-\phi^a(y,\delta,j)))\, a(y,\delta,j)\, \chi_1(y_1)\, \chi_1(y_2)\, \mathrm{d}y. \end{aligned} \tag{3.3.22}$$

Similarly as in the case $2^{2j}\delta_3 \gg 1$, we can consider either the substitution $(z_1,z_2) = (\lambda_1 y_1, \lambda_2 2^{-j} y_2)$, or the substitution $(z_1,z_2) = (\lambda_1 y_1, \lambda_3\phi^a(y,\delta,j))$ (in order to carry this out one needs to consider the cases $y_2 \sim 1$ and $y_2 \sim -1$ separately). Then one can easily obtain

$$\|\nu_j^\lambda\|_{L^\infty} \lesssim \min\{2^j\lambda_3, \lambda_2\}. \tag{3.3.23}$$

Next we calculate the L^∞ bounds on the Fourier transform by using the expression (3.3.20).

Case 1. $\lambda_1 \ll \lambda_2$ or $\lambda_1 \gg \lambda_2$, and $\lambda_3 \ll \max\{\lambda_1,\lambda_2\}$. By integration by parts in x_1 one has

$$\|\widehat{\nu_j^\lambda}\|_{L^\infty} \lesssim \Big(\max\{\lambda_1,\lambda_2\}\Big)^{-N}.$$

The operators T_j^λ are now summable which can be seen by using the estimate in (3.3.6) obtained by interpolation.

Case 2. $\lambda_1 \ll \lambda_2$ or $\lambda_1 \gg \lambda_2$, and $\lambda_3 \gtrsim \max\{\lambda_1,\lambda_2\}$. Here we use integration by parts in x_2 only and so we have the bound

$$\|\widehat{\nu_j^\lambda}\|_{L^\infty} \lesssim \lambda_3^{-N}.$$

After interpolating we can again sum operators T_j^λ in all three paramteres.

Case 3. $\lambda_1 \sim \lambda_2$ and $\lambda_3 \ll 2^{-j}\lambda_2$. Note that necessarily $\lambda_2 \geqslant 2^j$. Here we use stationary phase in x_1 and integration by parts in x_2. Then one gets the estimate

$$\|\widehat{\nu_j^\lambda}\|_{L^\infty} \lesssim \lambda_1^{-1/2}(2^{-j}\lambda_2)^{-N}.$$

The better bound in (3.3.23) is $2^j\lambda_3$. Therefore (3.3.6) becomes

$$\|T_j^\lambda\|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)}) \to L^{\mathfrak{p}_3'}_{x_3}(L^{\mathfrak{p}_1'}_{(x_1,x_2)})} \lesssim (\lambda_1^{-1/2}(2^{-j}\lambda_2)^{-N})^{1-\theta}(2^j\lambda_3)^\theta \lambda_3^{\frac{1}{2}-\theta}.$$

If $\theta < 1/3$, then we can rewrite

$$\|T_j^\lambda\|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)}) \to L^{\mathfrak{p}_3'}_{x_3}(L^{\mathfrak{p}_1'}_{(x_1,x_2)})} \lesssim (\lambda_1^{-1/2}\lambda_3^{-N})^{1-\theta}\lambda_1^\theta \lambda_3^{\frac{1}{2}-\theta},$$

we note that one can now easily sum in both λ_1 and λ_3. If $\theta = 1/3$, then the first inequality for T_j^λ can be rewritten as

$$\|T_j^\lambda\|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)}) \to L^{\mathfrak{p}_3'}_{x_3}(L^{\mathfrak{p}_1'}_{(x_1,x_2)})} \lesssim (2^{-j}\lambda_1)^{-N}\lambda_3^{1/2},$$

for some different N. Now we first sum in λ_3 up to $2^{-j}\lambda_1$, and then we sum in $\lambda_1 \geqslant 2^j$.

Case 4. $\lambda_1 \sim \lambda_2$ and $\lambda_3 \sim 2^{-j}\lambda_2$. Again necessarily $\lambda_2 \gtrsim 2^j$. One uses in both x_1 and x_2 the stationary phase method and gets

$$\|\widehat{\nu_j^\lambda}\|_{L^\infty} \lesssim 2^{j/2}\lambda_1^{-1}.$$

The estimate for $\|\nu_j^\lambda\|_{L^\infty}$ from (3.3.23) is $\lesssim \lambda_2$. Hence, we get the estimate

$$\begin{aligned}\|T_j^\lambda\|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})\to L^{\mathfrak{p}_3'}_{x_3}(L^{\mathfrak{p}_1'}_{(x_1,x_2)})} &\lesssim (2^{j/2}\lambda_1^{-1})^{1-\theta}\lambda_1^\theta\lambda_3^{1/2-\theta}\\ &\lesssim 2^{j\theta/2}\lambda_1^{\theta-1/2}.\end{aligned}$$

By summation in $\lambda_1 \gtrsim 2^j$ we obtain the bound

$$2^{3j\theta/2-j/2}.$$

Now since $\theta \leqslant 1/3$, we get the desired result.

Case 5. $\lambda_1 \sim \lambda_2$ and $\lambda_3 \gtrsim \lambda_2$. Here it suffices to use integration by parts in x_2 only. One easily gets

$$\|T_j^\lambda\|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})\to L^{\mathfrak{p}_3'}_{x_3}(L^{\mathfrak{p}_1'}_{(x_1,x_2)})} \lesssim \lambda_3^{-N},$$

and one can now sum in both λ_1 and λ_3.

Case 6. $\lambda_1 \sim \lambda_2$ and $2^{-j}\lambda_2 \ll \lambda_3 \ll \lambda_2$. By the stationary phase method in x_1 and integration by parts in x_2

$$\|\widehat{\nu_j^\lambda}\|_{L^\infty} \lesssim \lambda_1^{-\frac{1}{2}}(\lambda_3)^{-N},$$

and the better bound in (3.3.23) is λ_2.

Similarly as in the case $2^{2j}\delta_3 \gg 1$ one easily sees that, unless $\theta = 1/3$, one can sum in both parameters. Henceforth we shall assume $\theta = 1/3$ and use complex interpolation in order to deal with this case. Here we know that ϕ has A type singularity and $\tilde{\sigma} = 1/4$. For simplicity we shall again assume that $\lambda_1 = \lambda_2$.

We consider the following function parametrized by the complex number ζ and the dyadic number λ_3:

$$\mu_\zeta^{\lambda_3} = \gamma(\zeta) \sum_{\lambda_3 \ll \lambda_1 \ll 2^j\lambda_3} (\lambda_1)^{\frac{1-3\zeta}{2}} \nu_j^\lambda,$$

where

$$\gamma(\zeta) = 2^{-3(\zeta-1)/2} - 1.$$

We denote the associated convolution operator by $T_\zeta^{\lambda_3}$. For $\zeta = 1/3$ we see that

$$\mu_\zeta^{\lambda_3} = \sum_{\lambda_3 \ll \lambda_1 \ll 2^j\lambda_3} \nu_j^\lambda.$$

Hence, by interpolation it suffices to prove

$$\|T^{\lambda_3}_{it}\|_{L^{2/(2-\tilde{\sigma})}_{x_3}(L^1_{(x_1,x_2)})\to L^{2/\tilde{\sigma}}_{x_3}(L^\infty_{(x_1,x_2)})} \lesssim \lambda_3^{-N},$$
$$\|T^{\lambda_3}_{1+it}\|_{L^2\to L^2} \lesssim 1,$$

for some $N > 0$, with constants uniform in $t \in \mathbb{R}$.

The first estimate follows right away since $\widehat{\nu^\lambda_j}$ have essentially disjoint supports, and so the L^∞ estimate for $\widehat{\nu^\lambda_j}$ implies

$$\|\widehat{\mu^{\lambda_3}_{it}}\|_{L^\infty} \lesssim \lambda_3^{-N},$$

for any $N \in \mathbb{N}$.

We prove the second estimate using Lemma 2.2.5. We need to prove

$$\Big\| \sum_{\lambda_3 \ll \lambda_1 \ll 2^j\lambda_3} (\lambda_1)^{-1-\frac{3}{2}it}\, \nu^\lambda_j \Big\|_{L^\infty} \lesssim \frac{1}{\left|2^{-\frac{3}{2}it}-1\right|}, \tag{3.3.24}$$

uniformly in t.

We first use the substitution $(z_1,z_2) = (y_1, \phi^a(y_1,y_2,\delta,j))$ in the expression (3.3.22), considering the cases $y_2 \sim 1$ and $y_2 \sim -1$ separately. In order to solve for (y_1,y_2) in terms of (z_1,z_2), we introduce for a moment intermediary coordinates $(\tilde{y}_1,\tilde{y}_2) = (y_1, 2^{-j}y_2 + y_1^m\omega(\delta_1 y_1))$. In coordinates $(\tilde{y}_1,\tilde{y}_2)$ the expression for $\phi^a = z_2$ becomes

$$2^{2j}\tilde{b}(\tilde{y}_1,\tilde{y}_2,\delta_1,\delta_2)(\tilde{y}_2 - \tilde{y}_1^m\omega(\delta_1\tilde{y}_1))^2 + 2^{2j}\delta_3\tilde{y}_1^n\beta(\delta_1\tilde{y}_1).$$

Then one can easily see that by solving for $\tilde{y}_2$ in terms of (z_1,z_2), one gets precisely the expression (3.3.9) as in the case $2^{2j}\delta_3 \gg 1$. Therefore by solving for y_2 in terms of (z_1,z_2) one gets

$$y_2 = \pm\tilde{b}_1\Big(z_1, \sqrt{2^{-2j}z_2 - \delta_3 z_1^n\beta(\delta_1 z_1)}, \delta_1, \delta_2\Big) \times \sqrt{z_2 - 2^{2j}\delta_3 z_1^n\beta(\delta_1 z_1)},$$

where now both z_1 and z_2 are positive. We shall from now on consider y_2 as a function of (z_1,z_2). On the limit $j \to \infty$ and $\delta \to 0$ the function $y_2 = y_2(z_1,z_2,\delta,j)$ converges to $\pm C\sqrt{z_2}$ for some constant $C \neq 0$ since we are in the $\theta = 1/3$ case (i.e., A type singularity case); see (3.3.10).

After applying the just introduced substitution to the expression (3.3.22) we get

$$\begin{aligned}\nu^\lambda_j(x) = \lambda_1\lambda_2\lambda_3 \int & \check{\chi}_1(\lambda_1(x_1-z_1))\,\check{\chi}_1(\lambda_2(x_2 - 2^{-j}y_2(z_1,z_2,\delta,j) - z_1^m\omega(\delta_1 z_1)))\\ &\times \check{\chi}_1(\lambda_3(x_3-z_2))\,\tilde{a}_1(z,\delta,j)\,\chi_1(z_1)\,\chi_1(y_2(z_1,z_2,\delta,j))\,\mathrm{d}z,\end{aligned}$$

where $\tilde{a}_1$ is the function a multiplied by the Jacobian of the change of variables. Since $|y_2| \sim 1$ is equivalent to $|z_2| \sim 1$, we may rewrite again the above expression as

$$\begin{aligned}\nu^\lambda_j(x) = \lambda_1\lambda_2\lambda_3 \int & \check{\chi}_1(\lambda_1(x_1-z_1))\\ &\times \check{\chi}_1(\lambda_2(x_2 - 2^{-j}y_2(z_1,z_2,\delta,j) - z_1^m\omega(\delta_1 z_1)))\\ &\times \check{\chi}_1(\lambda_3(x_3-z_2))\,\tilde{a}(z,\delta,j)\,\chi_1(z_1)\,\chi_1(z_2)\,\mathrm{d}z.\end{aligned} \tag{3.3.25}$$

Now we substitute $\lambda_1 z_1 \mapsto z_1$ and $\lambda_3 z_2 \mapsto z_2$ in the expression (3.3.25), plug it into the sum (3.3.24), and obtain

$$\begin{aligned}\sum_{\lambda_3 \ll \lambda_1 \ll 2^j \lambda_3} (\lambda_1)^{-\frac{3}{2}it} \int & \check{\chi}_1(\lambda_1 x_1 - z_1) \\ & \times \check{\chi}_1(\lambda_1 x_2 - 2^{-j}\lambda_1 y_2(\lambda_1^{-1}z_1, \lambda_3^{-1}z_2, \delta, j) - \lambda_1^{-m+1} z_1^m \omega(\delta_1 \lambda_1^{-1} z_1)) \\ & \times \check{\chi}_1(\lambda_3 x_3 - z_2) \times \tilde{a}(\lambda_1^{-1}z_1, \lambda_3^{-1}z_2, \delta, 2^{-j})\, \chi_1(\lambda_1^{-1}z_1)\, \chi_1(\lambda_3^{-1}z_2)\, \mathrm{d}z.\end{aligned}$$

Now we have $z_1 \sim \lambda_1$, $z_2 \sim \lambda_3$, and $|y_2(\lambda_1^{-1}z_1, \lambda_3^{-1}z_2, \delta, j)| \sim 1$.

We can assume $|(x_1, x_2)| \leqslant C$ for some large constant C, since otherwise we can use the first two factors within the integral and gain a factor of λ_1^{-N}. Similarly as in the case $2^{2j}\delta_3 \gg 1$ we shall consider integration in z_1 only (and $\lambda_3^{-1}z_2$ shall be a bounded parameter), and one can also use the substitution $z_1 \mapsto \lambda_1 x_1 - z_1$ to reduce the problem to when $|z_1| \ll \lambda_1^{\varepsilon}$ and $x_1 \sim 1$. We also introduce $\psi_\delta(x_1) = x_1^m \omega(\delta_1 x_1)$. Then it remains to estimate

$$\begin{aligned}\sum_{\lambda_3 \ll \lambda_1 \ll 2^j \lambda_3} (\lambda_1)^{-\frac{3}{2}it} \int & \check{\chi}_1(z_1) \check{\chi}_1(\lambda_1(x_2 - \psi_\delta(x_1 - \lambda_1^{-1}z_1) - 2^{-j}\lambda_1 y_2(x_1 - \lambda_1^{-1}z_1, \lambda_3^{-1}z_2, \delta, j))) \\ & \times \tilde{a}(x_1 - \lambda_1^{-1}z_1, \lambda_3^{-1}z_2, \delta, 2^{-j})\, \chi_1(x_1 - \lambda_1^{-1}z_1)\, \chi_0(z_1 \lambda_1^{-\varepsilon})\, \mathrm{d}z_1.\end{aligned}$$

Within the second factor in the integral we can use a Taylor approximation at x_1 and obtain

$$\begin{aligned}\sum_{\lambda_3 \ll \lambda_1 \ll 2^j \lambda_3} (\lambda_1)^{-\frac{3}{2}it} \int & \check{\chi}_1(z_1) \check{\chi}_1(\lambda_1 Q(x_1, x_2, \lambda_3^{-1}z_2, \delta, 2^{-j}) + z_1 r(\lambda_1^{-1}z_1, x_1, \lambda_3^{-1}z_2, \delta, 2^{-j})) \\ & \times \tilde{a}(x_1 - \lambda_1^{-1}z_1, \lambda_3^{-1}z_2, \delta, 2^{-j})\, \chi_1(x_1 - \lambda_1^{-1}z_1)\, \chi_0(z_1 \lambda_1^{-\varepsilon})\, \mathrm{d}z_1,\end{aligned}$$

where $|\partial_1^N r| \sim 1$ for $N \geqslant 0$ since the term ψ_δ is dominant, and Q is a smooth function with uniform bounds. Now we notice that this form is the same as in the case $2^{2j}\delta_3 \gg 1$ in the part where we used complex interpolation, and hence the same proof using the oscillatory sum lemma can be applied, up to obvious changes such as changing the summation bounds.

3.3.6 The case $2^{2j}\delta_3 \sim 1$

As in [51] we denote

$$\sigma := 2^{2j}\delta_3, \qquad b^\#(x, \delta, j) := \tilde{b}(x_1, 2^{-j}x_2 + x_1^m \omega(\delta_1 x_1), \delta),$$

and so $\sigma \sim 1$ and $|b^\#(x, \delta, j)| \sim 1$. Therefore the complete phase can be rewritten as

$$\begin{aligned}\Phi(x, \delta, j, \xi) :=& \xi_1 x_1 + \xi_2 x_1^m \omega(\delta_1 x_1) + \xi_3 \sigma x_1^n \beta(\delta_1 x_1) \\ & + 2^{-j}\xi_2 x_2 + \xi_3 b^\#(x, \delta, j) x_2^2.\end{aligned} \tag{3.3.26}$$

Recall also that in this case we have the weaker conditions $x_1 \sim 1$ and $|x_2| \lesssim 1$ for the domain of integration in the integral in (3.3.20).

We furthermore slightly modify the notation in this case, as it was done in [51]. Namely, δ shall denote in this subsection (δ_1, δ_2) since δ_3 appears only in σ. We also note that in this case there is no A_∞ nor D_∞ type singularity.

Let us introduce the notation

$$\psi_\omega(y_1) = y_1^m \omega(\delta_1 y_1), \qquad \psi_\beta(y_1) = \sigma y_1^n \beta(\delta_1 y_1).$$

Then, after applying the inverse Fourier transform to (3.3.20), we may write

$$\begin{aligned} \nu_j^\lambda(x) = \lambda_1\lambda_2\lambda_3 \int & \check{\chi}_1(\lambda_1(x_1 - y_1))\, \check{\chi}_1(\lambda_2(x_2 - 2^{-j}y_2 - \psi_\omega(y_1))) \\ & \times \check{\chi}_1(\lambda_3(x_3 - b^\#(y,\delta,j)y_2^2 - \psi_\beta(y_1))) \\ & \times a(y,\delta,j)\, \chi_1(y_1)\, \chi_0(y_2)\, \mathrm{d}y. \end{aligned} \tag{3.3.27}$$

As was noted in [51, Subsection 4.2.2.], here we have the bounds

$$\|\nu_j^\lambda\|_{L^\infty} \lesssim \lambda_3^{1/2} \min\{2^j \lambda_3^{1/2}, \lambda_2\}. \tag{3.3.28}$$

Namely, in the first factor within the integral in (3.3.27) we can substitute $\lambda_1 y_1 \mapsto y_1$, and afterwards either substitute $\lambda_2 2^{-j} y_2 \mapsto y_2$ in the second factor, or use the van der Corput lemma (i.e., Lemma 2.2.1, (i)) in the third factor with respect to the y_2 variable.

As can easily be seen from (3.3.26) by using integration by parts in x_1, if one of λ_1, λ_2 is considerably larger than any other $\lambda_i, i = 1,2,3$, then we can easily gain a sufficiently strong estimate with which one can sum absolutely in all three parameters $\lambda_i, i = 1,2,3$, the operators T_j^λ.

If λ_3 is significantly larger than both λ_1 and λ_2 and ϕ is of type A, we can also use integration by parts in x_1 in order to get a sufficiently strong estimate. In the case when λ_3 is the largest and ϕ is of type D, then $b^\#(x,\delta,j)$ is approximately x_1 in the C^∞ sense, and so in this case and when $|x_2| \sim 1$, we use integration by parts in x_2, and when $|x_2| \ll 1$ integration by parts in x_1. In both parts we get the bound λ_3^{-N} with which we can obtain a summable estimate for T_j^λ in all three parameters.

As it turns out, in almost all the other possible relations between λ_i, $i = 1,2,3$, we shall need complex interpolation if $\theta = 1/3$, or if $\theta = 1/4$ and it is the "diagonal" case, i.e., all the λ_i, $i = 1,2,3$, are of approximately the same size. If $\theta = 1/3$ and λ_i, $i = 1,2,3$, are of approximately the same size we shall actually need a finer analysis where estimates on Airy integrals are needed. This will be done in the next section.

Case 1.1. $\lambda_1 \sim \lambda_3$, $\lambda_2 \ll \lambda_1$, and $\lambda_2 \leqslant 2^j \lambda_1^{1/2}$. On the part where $|x_2| \sim 1$ we can use integration by parts in x_2 and obtain much stronger estimates sufficient for absolute summation. When $|x_2| \ll 1$ we use stationary phase in both variables, and so

$$\|\widehat{\nu_j^\lambda}\|_{L^\infty} \lesssim \lambda_1^{-1}, \qquad \|\nu_j^\lambda\|_{L^\infty} \lesssim \lambda_1^{1/2}\lambda_2,$$

from which one can calculate that

$$\|T_j^\lambda\|_{L_{x_3}^{\mathfrak{p}_3}(L_{(x_1,x_2)}^{\mathfrak{p}_1}) \to L_{x_3}^{\mathfrak{p}_3'}(L_{(x_1,x_2)}^{\mathfrak{p}_1'})} \lesssim \lambda_1^{(\theta-1)/2} \lambda_2^\theta.$$

Let us denote by $T^I_{\delta,j}$ the sum of the operator pieces T^λ_j in this case. We need to separate the sum in λ_1 into two subcases $\lambda_1 \leqslant 2^{2j}$ and $\lambda_1 > 2^{2j}$:

$$\begin{aligned}\|T^I_{\delta,j}\|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})\to L^{\mathfrak{p}_3'}_{x_3}(L^{\mathfrak{p}_1'}_{(x_1,x_2)})} &\lesssim \sum_{\lambda_1=1}^{2^{2j}}\sum_{\lambda_2=1}^{\lambda_1}\lambda_1^{(\theta-1)/2}\lambda_2^\theta + \sum_{\lambda_1=2^{2j+1}}^{\infty}\sum_{\lambda_2=1}^{2^j\lambda_1^{1/2}}\lambda_1^{(\theta-1)/2}\lambda_2^\theta \\ &\lesssim \sum_{\lambda_1=1}^{2^{2j}}\lambda_1^{(3\theta-1)/2} + \sum_{\lambda_1=2^{2j+1}}^{\infty}2^{j\theta}\lambda_1^{(2\theta-1)/2} \\ &\lesssim \sum_{\lambda_1=1}^{2^{2j}}\lambda_1^{(3\theta-1)/2} + 2^{j(3\theta-1)}.\end{aligned}$$

Therefore if $\theta < 1/3$, then we obtain the desired result, and if $\theta = 1/3$, we need to use complex interpolation for the first sum where $\lambda_1 \leqslant 2^{2j}$. For $\theta = 1/3$, we have

$$\|T^\lambda_j\|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})\to L^{\mathfrak{p}_3'}_{x_3}(L^{\mathfrak{p}_1'}_{(x_1,x_2)})} \lesssim (\lambda_1\lambda_2^{-1})^{-1/3},$$

and one is easily convinced that we may restrict ourselves to the case

$$1 \ll \lambda_1 \ll 2^{2j}, \qquad 1 \ll \lambda_2 \ll \lambda_1.$$

The bound on the operator norm motivates us to define k through $2^k := \lambda_1\lambda_2^{-1} = 2^{k_1-k_2}$, where $2^{k_1} = \lambda_1$ and $2^{k_2} = \lambda_2$. Our goal is to prove that for each k within the range $1 \ll 2^k \ll 2^{2j}$ we have

$$\Big\|\sum_{\lambda_1\lambda_2^{-1}=2^k} T^\lambda_j\Big\|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})\to L^{\mathfrak{p}_3'}_{x_3}(L^{\mathfrak{p}_1'}_{(x_1,x_2)})} \lesssim 2^{-k/3},$$

since then we obtain the desired estimate by summation in k.

We shall slightly simplify the proof by assuming that $\lambda_1 = \lambda_3$. Let us consider the following function parametrized by the complex number ζ and the integer k:

$$\mu^k_\zeta = 2^{k\frac{3\zeta-1}{2}}\gamma(\zeta)\sum_{\lambda_1\lambda_2^{-1}=2^k}(\lambda_1)^{\frac{3-9\zeta}{4}}\nu^\lambda_j,$$

where

$$\gamma(\zeta) = \frac{2^{-9(\zeta-1)/4}-1}{2^{\frac{3}{2}}-1}.$$

The associated convolution operator (convolution against the Fourier transform of μ^k_ζ) we denote by T^k_ζ. For $\zeta = 1/3$ we see that

$$\mu^k_\zeta = \sum_{\lambda_1\lambda_2^{-1}=2^k}\nu^\lambda_j.$$

Therefore, it is sufficient to prove

$$\begin{aligned} \|T_{it}^k\|_{L^{2/(2-\tilde{\sigma})}_{x_3}(L^1_{(x_1,x_2)}) \to L^{2/\tilde{\sigma}}_{x_3}(L^\infty_{(x_1,x_2)})} &\lesssim 2^{-k/2}, \\ \|T_{1+it}^k\|_{L^2 \to L^2} &\lesssim 1, \end{aligned}$$

with constants uniform in $t \in \mathbb{R}$. Recall that $\tilde{\sigma} = 1/4$ since $m = 2$ and $\theta = 1/3$.

The first estimate follows right away. Namely, since $\widehat{\nu_j^\lambda}$ have supports located at λ, then by the estimate for the L^∞ norm of the function $\widehat{\nu_j^\lambda}$ we have

$$|\widehat{\mu_{it}^k}(\xi)| \lesssim \frac{2^{-k/2}}{(1+|\xi_3|)^{1/4}},$$

and now one needs to recall Lemma 2.3.1.

We prove the second estimate by using Lemma 2.2.5. We need to prove

$$\begin{aligned} &\Big\| \sum_{\lambda_1 \lambda_2^{-1} = 2^k} 2^k (\lambda_1)^{-\frac{3}{2} - \frac{9}{4}it} \nu_j^\lambda \Big\|_{L^\infty} \\ &\qquad = \Big\| \sum_{2^k \ll \lambda_1 \ll 2^{2j}} \lambda_1^{-1/2} \lambda_2^{-1} \lambda_1^{-\frac{9}{4}it} \nu_j^{(\lambda_1, \lambda_1 2^{-k}, \lambda_1)} \Big\|_{L^\infty} \lesssim \frac{1}{\left|2^{-\frac{9}{4}it} - 1\right|}, \end{aligned} \tag{3.3.29}$$

uniformly in t.

After substituting $\lambda_1 y_1 \mapsto y_1$ and $\lambda_1^{1/2} y_2 \mapsto y_2$ in the expression (3.3.27), we get that the sum on the left hand side of (3.3.29) is

$$\begin{aligned} \sum_{2^k \ll \lambda_1 \ll 2^{2j}} \lambda_1^{-\frac{9}{4}it} \int & \check{\chi}_1(\lambda_1 x_1 - y_1)\, \check{\chi}_1(2^{-k}\lambda_1 x_2 - 2^{-j-k}\lambda_1^{1/2} y_2 - 2^{-k}\lambda_1 \psi_\omega(\lambda_1^{-1} y_1)) \\ & \times \check{\chi}_1(\lambda_1 x_3 - b^\#(\lambda_1^{-1} y_1, \lambda_1^{-1/2} y_2, \delta, j) y_2^2 - \lambda_1 \psi_\beta(\lambda_1^{-1} y_1)) \\ & \times a(\lambda_1^{-1} y_1, \lambda_1^{-1/2} y_2, \delta, j)\, \chi_1(\lambda_1^{-1} y_1)\, \chi_0(\lambda_1^{-1/2} y_2)\, \mathrm{d}y. \end{aligned}$$

Using the first three factors we can reduce the problem to the case $|x| \leqslant C$ for some large constant C. Now, as we have done in previous instances of complex interpolation, we use the substitution $\lambda_1 x_1 - y_1 \mapsto y_1$, conclude that it is sufficient to consider the part of the integration domain where $|y_1| \leqslant \lambda_1^\varepsilon$. In particular then $x_1 \sim 1$ and we can use Taylor approximation for ψ_ω and ψ_β at x_1. Then one gets

$$\begin{aligned} \sum_{2^k \ll \lambda_1 \leqslant 2^{2j}} \lambda_1^{-\frac{9}{4}it} \int & \check{\chi}_1(y_1)\, \check{\chi}_1(2^{-k}\lambda_1 Q_\omega(x_1, x_2, \delta_1) - 2^{-k} y_1 r_\omega(\lambda_1^{-1} y_1, x_1, \delta_1) - 2^{-j-k}\lambda_1^{1/2} y_2) \\ & \times \check{\chi}_1(\lambda_1 Q_\beta(x_1, x_3, \delta_1) - y_1 r_\beta(\lambda_1^{-1} y_1, x_1, \delta_1) - b^\#(x_1 - \lambda_1^{-1} y_1, \lambda_1^{-1/2} y_2, \delta, j) y_2^2) \\ & \times a(x_1 - \lambda_1^{-1} y_1, \lambda_1^{-1/2} y_2, \delta, j)\, \chi_1(x_1 - \lambda_1^{-1} y_1)\, \chi_0(\lambda_1^{-1/2} y_2)\, \chi_0(\lambda_1^{-\varepsilon} y_1)\, \mathrm{d}y, \end{aligned}$$

where $|\partial_1^N r_\omega| \sim 1$ and $|\partial_1^N r_\beta| \sim 1$ for any $N \geqslant 0$. Also note that $2^{-j}\lambda_1^{1/2} \ll 1$.

We may now conclude that it is sufficient to consider the cases when either $|A| \gg 1$ or $|B| \gg 1$, where

$$A := 2^{-k}\lambda_1 Q_\omega(x_1, x_2, \delta_1), \qquad\qquad B := \lambda_1 Q_\beta(x_1, x_3, \delta_1),$$

since otherwise, when both $|A|$ and $|B|$ are bounded, we could apply Lemma 2.2.5, similarly as in the case $2^{2j}\delta_3 \gg 1$, to the function

$$\begin{aligned} H(z_1, z_2, &z_3, z_4, z_5; x, \delta, \sigma) := \\ &\int \check{\chi}_1(y_1)\, \check{\chi}_1(z_1 - 2^{-k} y_1 r_\omega(z_4^{1/\varepsilon} y_1, x_1, \delta_1) - 2^{-k} z_3 y_2) \\ &\times \check{\chi}_1(z_2 - y_1 r_\beta(z_4^{1/\varepsilon} y_1, x_1, \delta_1) - b^\#(x_1 - z_4^{1/\varepsilon} y_1, z_4^{1/(2\varepsilon)} y_2, \delta, j) y_2^2) \\ &\times a(x_1 - z_4^{1/\varepsilon} y_1, z_4^{1/(2\varepsilon)} y_2, \delta, j)\, \chi_1(x_1 - z_4^{1/\varepsilon} y_1)\, \chi_0(z_4^{1/(2\varepsilon)} y_2)\, \chi_0(z_4 y_1)\, \mathrm{d}y, \end{aligned}$$

where we would plug in

$$(z_1, z_2, z_3, z_4, z_5) = (2^{-k}\lambda_1 Q_\omega(x_1, x_2, \delta_1), \lambda_1 Q_\beta(x_1, x_3, \delta_1), 2^{-j}\lambda_1^{1/2}, \lambda_1^{-\varepsilon}, 2^k \lambda_1^{-1}).$$

Note that the upper bounds on z_4 and z_5 are given by the summation bounds for the parameter λ_1, and that the function H does not depend on z_5. Furthermore, the C^1 norm of H in $(z_1, z_2, z_3, z_4, z_5)$ is bounded since derivatives of Schwartz functions are Schwartz and only factors of polynomial growth in y_1 and y_2 appear when taking the derivatives. The polynomial growth in y_1 can be dealt with by using the first factor. For the polynomial growth in y_2 one has to consider the cases $|y_2| \lesssim |y_1|^N$ and $|y_2| \gg |y_1|^N$ separately. In the first case we can obviously again use the first factor, and in the second case we use the third factor inside which the term $b^\# y_2^2$ is now dominant.

Let us now first assume $|B| \gg 1$. The first three factors within the integral are behaving essentially like

$$\check{\chi}_1(y_1)\check{\chi}_1(A - 2^{-k} y_1 - 2^{-j-k}\lambda_1^{1/2} y_2)\check{\chi}_1(B - y_1 - y_2^2).$$

We may reduce ourselves to the discussion of the part of the integration domain where $|y_1| \ll |B|^{\varepsilon_B}$ since otherwise, when $|y_1| \gtrsim |B|^{\varepsilon_B}$, we could use the first factor, obtain the estimate $|B|^{-N\varepsilon_B}$ for the integral, and then sum this geometric series in λ_1. Then $|B - y_1 r_\beta| \sim |B|$, and the integral we need to estimate is bounded by

$$\int |\check{\chi}_1(B - y_1 r_\beta - y_2^2 b^\#)| \mathrm{d}y_2 \lesssim \int |\check{\chi}_1(B - y_1 r_\beta - t)|\, |t|^{-1/2} \mathrm{d}t \leqslant C|B|^{-1/2},$$

for some constant C. Now one can again sum in λ_1.

Let us now assume $|B| \leqslant C_B$ for some large, but fixed constant C_B, and let $|A| \gg C_B$. Again, we can reduce ourselves to the part where $|y_1| \ll |A|^{\varepsilon_A}$, and so $|A - 2^{-k} y_1 r_\omega| \sim |A|$. Therefore if $|y_2| \leqslant |A|^{1/2}$, then using the second factor we get that the integral is bounded (up to a constant) by $|A|^{-N}$. If $|y_2| > |A|^{1/2}$, then $|B - y_1 r_\beta - y_2^2 b^\#| \gtrsim |A|$ and so we can use the third factor, and sum in λ_1.

Case 1.2. $\lambda_1 \sim \lambda_3$, $\lambda_2 \ll \lambda_1$, and $\lambda_2 > 2^j \lambda_1^{1/2}$. In this case we have the same bound for the Fourier transform. Hence

$$\|\widehat{\nu_j^\lambda}\|_{L^\infty} \lesssim \lambda_1^{-1}, \qquad\qquad \|\nu_j^\lambda\|_{L^\infty} \lesssim 2^j \lambda_1,$$

from which one can calculate that

$$\|T_j^\lambda\|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})\to L^{\mathfrak{p}_3'}_{x_3}(L^{\mathfrak{p}_1'}_{(x_1,x_2)})} \lesssim \lambda_1^{\theta-1/2} 2^{j\theta}.$$

If we denote by $T^{II}_{\delta,j}$ the sum of the operator pieces in this case, then we have:

$$\begin{aligned}\|T^{II}_{\delta,j}\|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})\to L^{\mathfrak{p}_3'}_{x_3}(L^{\mathfrak{p}_1'}_{(x_1,x_2)})} &\lesssim 2^{j\theta} \sum_{\lambda_1=2^{2j}}^{\infty} \sum_{\lambda_2=2^j\lambda_1^{1/2}}^{\lambda_1} \lambda_1^{\theta-1/2} \\ &\lesssim 2^{j\theta} \sum_{\lambda_1=2^{2j}}^{\infty} (\log_2 \lambda_1 - 2j)\lambda_1^{\theta-1/2} \\ &\lesssim 2^{j(3\theta-1)} \lesssim 1.\end{aligned}$$

Case 2.1. $\lambda_2 \sim \lambda_3$, $\lambda_1 \ll \lambda_2$, and $\lambda_2 \leqslant 2^{2j}$. Here again we may use stationary phase in both variables (and when $|x_2| \sim 1$ even integration by parts in x_2). The estimates are

$$\|\widehat{\nu_j^\lambda}\|_{L^\infty} \lesssim \lambda_2^{-1}, \qquad \|\nu_j^\lambda\|_{L^\infty} \lesssim \lambda_2^{3/2},$$

and therefore independent of λ_1. As in [51] we define

$$\sigma_j^{\lambda_2,\lambda_3} = \sum_{\lambda_1 \ll \lambda_2} \nu_j^\lambda,$$

and note that then we can write

$$\widehat{\sigma_j^{\lambda_2,\lambda_3}} = \chi_0\Big(\frac{\xi_1}{\lambda_2}\Big)\chi_1\Big(\frac{\xi_2}{\lambda_2}\Big)\chi_1\Big(\frac{\xi_3}{\lambda_3}\Big)\widehat{\nu}_{\delta,j},$$

where χ_0 is a smooth cutoff function supported in a sufficiently small neighbourhood of 0. Therefore, one easily sees that using the same argumentation as for ν_j^λ we have

$$\|\widehat{\sigma_j^{\lambda_2,\lambda_3}}\|_{L^\infty} \lesssim \lambda_2^{-1}, \qquad \|\sigma_j^{\lambda_2,\lambda_3}\|_{L^\infty} \lesssim \lambda_2^{3/2}.$$

The operator norm bound is

$$\|T_j^{\lambda_2,\lambda_3}\|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})\to L^{\mathfrak{p}_3'}_{x_3}(L^{\mathfrak{p}_1'}_{(x_1,x_2)})} \lesssim \lambda_2^{(3\theta-1)/2}.$$

Hence, if $\theta < 1/3$, then we obtain the desired result by summing the geometric series, and if $\theta = 1/3$, we need to use complex interpolation.

As usual, we consider only the case $\lambda_2 = \lambda_3$. Also note that we may reduce ourselves to the summation over $\lambda_2 \ll 2^{2j}$ instead of $\lambda_2 \leqslant 2^{2j}$. We define the following function parametrized by the complex number ζ:

$$\mu_\zeta = \gamma(\zeta) \sum_{1 \ll \lambda_2 \ll 2^{2j}} (\lambda_2)^{\frac{3-9\zeta}{4}} \sigma_j^{\lambda_2,\lambda_2},$$

where

$$\gamma(\zeta) = \frac{2^{-9(\zeta-1)/4} - 1}{2^{\frac{3}{2}} - 1}.$$

The associated convolution operator we denote by T_ζ. For $\zeta = 1/3$ we see that

$$\mu_\zeta = \sum_{1 \ll \lambda_2 \ll 2^{2j}} \sigma_j^{\lambda_2,\lambda_2},$$

and so it is sufficient to prove

$$\begin{aligned} \|T_{it}\|_{L^{2/(2-\tilde{\sigma})}_{x_3}(L^1_{(x_1,x_2)}) \to L^{2/\tilde{\sigma}}_{x_3}(L^\infty_{(x_1,x_2)})} &\lesssim 1, \\ \|T_{1+it}\|_{L^2 \to L^2} &\lesssim 1, \end{aligned}$$

with constants uniform in $t \in \mathbb{R}$.

The first estimate follows right away since $\sigma_j^{\lambda_2,\lambda_2}$ have ξ_3-supports located around λ_2, which implies by the L^∞ estimate for the Fourier transform of ν_j^λ that

$$|\widehat{\mu_{it}}(\xi)| \lesssim \frac{1}{(1 + |\xi_3|)^{1/4}}.$$

Now we can apply Lemma 2.3.1.

We prove the second estimate by using the oscillatory sum lemma (Lemma 2.2.5). We need to prove

$$\left\| \sum_{1 \ll \lambda_2 \ll 2^{2j}} (\lambda_1)^{-\frac{3}{2} - \frac{9}{4}it} \sigma_j^{\lambda_2,\lambda_2} \right\|_{L^\infty} \lesssim \frac{1}{\left| 2^{-\frac{9}{4}it} - 1 \right|}, \tag{3.3.30}$$

uniformly in t.

First note that since we obtain the function $\sigma_j^{\lambda_2,\lambda_2}$ by summation in λ_1, the expression (3.3.27) has to be replaced by

$$\begin{aligned} \sigma_j^{\lambda_2,\lambda_2} = \lambda_2^3 \int & \check{\chi}_0(\lambda_2(x_1 - y_1))\, \check{\chi}_1(\lambda_2(x_2 - 2^{-j} y_2 - \psi_\omega(y_1))) \\ & \times \check{\chi}_1(\lambda_2(x_3 - b^\#(y,\delta,j) y_2^2 - \psi_\beta(y_1))) \\ & \times a(y,\delta,j)\, \chi_1(y_1)\, \chi_0(y_2)\, \mathrm{d}y. \end{aligned} \tag{3.3.31}$$

Recall that the function χ_0 of the first factor within the integral has support contained in $[-\epsilon, \epsilon]$ where the small constant ϵ depends on the implicit constant in the relation $\lambda_1 \ll \lambda_2$.

After substituting $\lambda_2 y_1 \mapsto y_1$ and $\lambda_2^{1/2} y_2 \mapsto y_2$ in the expression (3.3.31), we get that the sum on the left hand side of (3.3.30) is

$$\begin{aligned} \sum_{1 \ll \lambda_2 \ll 2^{2j}} \lambda_2^{-\frac{9}{4}it} \int & \check{\chi}_1(\lambda_2 x_1 - y_1)\, \check{\chi}_1(\lambda_2 x_2 - 2^{-j} \lambda_2^{1/2} y_2 - \lambda_2 \psi_\omega(\lambda_2^{-1} y_1)) \\ & \times \check{\chi}_1(\lambda_2 x_3 - b^\#(\lambda_2^{-1} y_1, \lambda_2^{-1/2} y_2, \delta, j) y_2^2 - \lambda_2 \psi_\beta(\lambda_2^{-1} y_1)) \\ & \times a(\lambda_2^{-1} y_1, \lambda_2^{-1/2} y_2, \delta, j)\, \chi_1(\lambda_2^{-1} y_1)\, \chi_0(\lambda_2^{-1/2} y_2)\, \mathrm{d}y. \end{aligned}$$

Since otherwise we could use the first three factors within the integral to gain a factor of λ_2^{-N}, we may assume that $|x| \leqslant C$ for some large constant C.

Now again we use the substitution $\lambda_2 x_1 - y_1 \mapsto y_1$, conclude that it is sufficient to consider the part of the integration domain where $|y_1| \leqslant \lambda_2^{\varepsilon}$, which implies $x_1 \sim 1$, and so we may use Taylor approximation for ψ_ω and ψ_β at x_1. Then one gets

$$\begin{aligned}\sum_{1 \ll \lambda_2 \ll 2^{2j}} \lambda_2^{-\frac{9}{4}it} \int & \check{\chi}_1(y_1)\, \check{\chi}_1(\lambda_2 Q_\omega(x_1,x_2,\delta_1) - y_1 r_\omega(\lambda_2^{-1}y_1, x_1, \delta_1) - 2^{-j}\lambda_2^{1/2} y_2) \\ & \times \check{\chi}_1(\lambda_2 Q_\beta(x_1,x_3,\delta_1) - y_1 r_\beta(\lambda_2^{-1}y_1, x_1, \delta_1) - b^\#(x_1 - \lambda_2^{-1}y_1, \lambda_2^{-1/2}y_2, \delta, j) y_2^2) \\ & \times a(x_1 - \lambda_2^{-1}y_1, \lambda_2^{-1/2}y_2, \delta, j)\, \chi_1(x_1 - \lambda_2^{-1}y_1)\, \chi_0(\lambda_2^{-1/2}y_2)\, \chi_0(\lambda_2^{-\varepsilon}y_1)\, \mathrm{d}y,\end{aligned}$$

where $|\partial_1^N r_\omega| \sim 1$ and $|\partial_1^N r_\beta| \sim 1$ for any $N \geqslant 0$. Note that $2^{-j}\lambda_2^{1/2} \ll 1$.

Now we may restrict ourselves to cases when either $|A| \gg 1$ or $|B| \gg 1$, where

$$A := \lambda_2 Q_\omega(x_1,x_2,\delta_1), \qquad\qquad B := \lambda_2 Q_\beta(x_1,x_3,\delta_1),$$

since otherwise we could apply the oscillatory sum lemma similarly as in Case 1.1.

The first three factors within the integral are behaving essentially like

$$\check{\chi}_1(y_1)\check{\chi}_1(A - y_1 - 2^{-j}\lambda_2^{1/2}y_2)\check{\chi}_1(B - y_1 - y_2^2).$$

Let us first consider $|B| \gg 1$, as in Case 1.1. As usual, we may restrict ourselves to the part of the integration domain where $|y_1| \ll |B|^{\varepsilon_B}$. Therefore there we have $|B - y_1| \sim |B|$, and the integral is bounded by

$$\int |\check{\chi}_1(B - y_1 r_\beta - y_2^2 b^\#)| \mathrm{d}y_2 \lesssim \int |\check{\chi}_1(B - y_1 r_\beta - t)|\, |t|^{-1/2} \mathrm{d}t \leqslant C|B|^{-1/2},$$

for some constant C. Now one can sum in λ_2.

Let us now assume $|B| \leqslant C_B$ for some large, but fixed constant, and $|A| \gg C_B$. Again, we may consider only the part of the integration domain where $|y_1| \ll |A|^{\varepsilon_A}$, and so here we have $|A - y_1| \sim |A|$. Therefore, if $|y_2| \leqslant |A|^{1/2}$, then using the second factor we get that the integral is bounded (up to a constant) by $|A|^{-N}$. If $|y_2| > |A|^{1/2}$, then $|B - y_1 r_\beta - y_2^2 b^\#| \gtrsim |A|$ and so we can use the third factor to gain $|A|^{-N}$, and sum in λ_2.

Case 2.2. $\lambda_2 \sim \lambda_3$, $\lambda_1 \ll \lambda_2$, and $\lambda_2 > 2^{2j}$. As in the previous case we use

$$\sigma_j^{\lambda_2,\lambda_3} = \sum_{\lambda_1 \ll \lambda_2} \nu_j^{\lambda},$$

and note that in this case the bounds are

$$\|\widehat{\sigma_j^{\lambda_2,\lambda_3}}\|_{L^\infty} \lesssim \lambda_2^{-1}, \qquad\qquad \|\sigma_j^{\lambda_2,\lambda_3}\|_{L^\infty} \lesssim 2^j \lambda_2.$$

The operator norm bound is

$$\|T_j^{\lambda_2,\lambda_3}\|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)}) \to L^{\mathfrak{p}_3'}_{x_3}(L^{\mathfrak{p}_1'}_{(x_1,x_2)})} \lesssim 2^{j\theta}\lambda_2^{\theta-1/2}.$$

This is summable over $\lambda_2 > 2^{2j}$ for all $\theta \leqslant 1/3$.

Case 3.1. $\lambda_1 \sim \lambda_2$**,** $\lambda_3 \ll \lambda_1$**, and** $\lambda_3^{1/2} \gtrsim 2^{-j}\lambda_1$**.** In this case, by stationary phase in both variables, the estimates are

$$\|\widehat{\nu_j^\lambda}\|_{L^\infty} \lesssim \lambda_1^{-1/2}\lambda_3^{-1/2}, \qquad \|\nu_j^\lambda\|_{L^\infty} \lesssim \lambda_1\lambda_3^{1/2}, \tag{3.3.32}$$

from which one can calculate that

$$\|T_j^\lambda\|_{L_{x_3}^{\mathfrak{p}_3}(L_{(x_1,x_2)}^{\mathfrak{p}_1}) \to L_{x_3}^{\mathfrak{p}_3'}(L_{(x_1,x_2)}^{\mathfrak{p}_1'})} \lesssim \lambda_1^{(3\theta-1)/2}.$$

The sum of the operator pieces in this case we denote by $T_{\delta,j}^V$. Then

$$\begin{aligned}
\|T_{\delta,j}^V\|_{L_{x_3}^{\mathfrak{p}_3}(L_{(x_1,x_2)}^{\mathfrak{p}_1}) \to L_{x_3}^{\mathfrak{p}_3'}(L_{(x_1,x_2)}^{\mathfrak{p}_1'})} &\lesssim \sum_{\lambda_1=1}^{2^j}\sum_{\lambda_3=1}^{\lambda_1}\lambda_1^{(3\theta-1)/2} + \sum_{\lambda_1=2^j}^{2^{2j}}\sum_{\lambda_3=(2^{-j}\lambda_1)^2}^{\lambda_1}\lambda_1^{(3\theta-1)/2} \\
&\lesssim \sum_{\lambda_1=1}^{2^{2j}}\sum_{\lambda_3=1}^{\lambda_1}\lambda_1^{(3\theta-1)/2} \\
&\lesssim \sum_{\lambda_1=1}^{2^{2j}}\lambda_1^{(3\theta-1)/2}\log_2(\lambda_1).
\end{aligned}$$

This is summable if and only if $\theta < 1/3$. For $\theta = 1/3$ we see

$$\|T_j^\lambda\|_{L_{x_3}^{\mathfrak{p}_3}(L_{(x_1,x_2)}^{\mathfrak{p}_1}) \to L_{x_3}^{\mathfrak{p}_3'}(L_{(x_1,x_2)}^{\mathfrak{p}_1'})} \lesssim 1.$$

Therefore, in this case we shall need the oscillatory sum lemma with two parameters (Lemma 2.2.7) when applying complex interpolation.

As usual we assume $\lambda_1 = \lambda_2$. We consider the following function parametrized by the complex number ζ:

$$\mu_\zeta = \gamma(\zeta)\sum_{\lambda_1,\lambda_3}\lambda_1^{\frac{1-3\zeta}{2}}\lambda_3^{\frac{1-3\zeta}{4}}\nu_j^\lambda,$$

where $\gamma(\zeta)$ is to be defined later as appropriate. The summation is over all λ_1 and λ_3 satisfying the conditions of this case (Case 3.1). Notice that we necessarily have $\lambda_1 \gg 1$.

We denote by T_ζ the associated convolution operator against the Fourier transform of μ_ζ. For $\zeta = 1/3$ we require that

$$\mu_\zeta = \sum_{\lambda_1,\lambda_3}\nu_j^\lambda,$$

i.e., $\gamma(1/3) = 1$. Then by interpolation it suffices to prove

$$\begin{aligned}
\|T_{it}\|_{L_{x_3}^{2/(2-\tilde{\sigma})}(L_{(x_1,x_2)}^1) \to L_{x_3}^{2/\tilde{\sigma}}(L_{(x_1,x_2)}^\infty)} &\lesssim 1, \\
\|T_{1+it}\|_{L^2 \to L^2} &\lesssim 1,
\end{aligned}$$

with constants uniform in $t \in \mathbb{R}$.

In order to prove the first estimate, we need the decay bound (2.3.2), i.e.,

$$|\widehat{\mu_{it}}(\xi)| \lesssim \frac{1}{(1+|\xi_3|)^{1/4}}.$$

But this follows automatically by (3.3.32), the definition of μ_ζ, and the fact that each $\widehat{\nu_j^\lambda}$ has its support located at λ.

It remains to prove the $L^2 \to L^2$ estimate by showing

$$\Big\| \sum_{\lambda_1,\lambda_3} (\lambda_1)^{-1-\frac{3}{2}it}(\lambda_3)^{-\frac{1}{2}-\frac{3}{4}it}\, \nu_j^\lambda \Big\|_{L^\infty} \lesssim \frac{1}{|\gamma(1+it)|}, \tag{3.3.33}$$

uniformly in t.

After substituting $\lambda_1 y_1 \mapsto y_1$ and $\lambda_3^{1/2} y_2 \mapsto y_2$ in the expression (3.3.27), we get that the sum on the left hand side of (3.3.33) is

$$\begin{aligned}\sum_{\lambda_1,\lambda_3} (\lambda_1)^{-\frac{3}{2}it}(\lambda_3)^{-\frac{3}{4}it} \int \check{\chi}_1(\lambda_1 x_1 - y_1)\, \check{\chi}_1(\lambda_1 x_2 - 2^{-j}\lambda_1\lambda_3^{-1/2} y_2 - \lambda_1 \psi_\omega(\lambda_1^{-1} y_1))&\\ \times \check{\chi}_1(\lambda_3 x_3 - b^\#(\lambda_1^{-1}y_1, \lambda_3^{-1/2}y_2, \delta, j) y_2^2 - \lambda_3 \psi_\beta(\lambda_1^{-1} y_1))&\\ \times a(\lambda_1^{-1}y_1, \lambda_3^{-1/2}y_2, \delta, j)\, \chi_1(\lambda_1^{-1}y_1)\, \chi_0(\lambda_3^{-1/2}y_2)\, \mathrm{d}y.&\end{aligned}$$

Using the first two factors we can restrict ourselves to the case when $|(x_1, x_2)| \leqslant C$ for some large constant C.

Next, we use the substitution $\lambda_1 x_1 - y_1 \mapsto y_1$, conclude that it is sufficient to consider integration over $|y_1| \leqslant \lambda_1^\varepsilon$ and that we have $x_1 \sim 1$. Then, after using the Taylor approximation for ψ_ω and ψ_β at x_1, one gets

$$\begin{aligned}&\sum_{\lambda_1,\lambda_3} (\lambda_1)^{-\frac{3}{2}it}(\lambda_3)^{-\frac{3}{4}it}\\ &\int \check{\chi}_1(y_1)\, \check{\chi}_1(\lambda_1 Q_\omega(x_1,x_2,\delta_1) - y_1 r_\omega(\lambda_1^{-1}y_1, x_1, \delta_1) - 2^{-j}\lambda_1\lambda_3^{-1/2} y_2)\\ &\times \check{\chi}_1(\lambda_3 Q_\beta(x_1,x_3,\delta_1) - \lambda_3\lambda_1^{-1} y_1 r_\beta(\lambda_1^{-1}y_1, x_1, \delta_1) - b^\#(x_1 - \lambda_1^{-1}y_1, \lambda_3^{-1/2}y_2, \delta, j) y_2^2)\\ &\times a(x_1 - \lambda_1^{-1}y_1, \lambda_3^{-1/2}y_2, \delta, j)\, \chi_1(x_1 - \lambda_1^{-1}y_1)\, \chi_0(\lambda_3^{-1/2}y_2)\, \chi_0(\lambda_1^{-\varepsilon}y_1)\, \mathrm{d}y,\end{aligned} \tag{3.3.34}$$

where $|\partial_1^N r_\omega| \sim 1$ and $|\partial_1^N r_\beta| \sim 1$ for any $N \geqslant 0$. Recall that $2^{-j}\lambda_1\lambda_3^{-1/2} \lesssim 1$ and $\lambda_3\lambda_1^{-1} \ll 1$.

If we define

$$A := \lambda_1 Q_\omega(x_1,x_2,\delta_1), \qquad B := \lambda_3 Q_\beta(x_1,x_3,\delta_1),$$

then we need to see what happens when either $|A| \gg 1$ or $|B| \gg 1$. Let us assume that C_B is a sufficiently large positive constant.

Subcase $|B| > C_B$ and $|A| \lesssim 1$. In this case we shall use the Hölder variant of the one parameter oscillatory sum lemma (Lemma 2.2.6) for each fixed λ_3. We define

$$
\begin{aligned}
&\tilde{H}(z_1, z_2, z_3, z_4; \lambda_3, x_1, x_3, \delta, 2^{-j}) \qquad (3.3.35)\\
&:= \int \check{\chi}_1(y_1)\, \check{\chi}_1(z_1 - y_1 r_\omega(z_3^{1/\varepsilon} y_1, x_1, \delta_1) - z_2 y_2)\\
&\quad \times \check{\chi}_1(\lambda_3 Q_\beta(x_1, x_3, \delta_1) - z_4 y_1 r_\beta(z_3^{1/\varepsilon} y_1, x_1, \delta_1) - b^{\#}(x_1 - z_3^{1/\varepsilon} y_1, \lambda_3^{-1/2} y_2, \delta, j) y_2^2)\\
&\quad \times a(x_1 - z_3^{1/\varepsilon} y_1, \lambda_3^{-1/2} y_2, \delta, j)\, \chi_1(x_1 - z_3^{1/\varepsilon} y_1)\, \chi_0(\lambda_3^{-1/2} y_2)\, \chi_0(z_3 y_1)\, \mathrm{d}y,
\end{aligned}
$$

where we shall plug in

$$
\begin{aligned}
z_1 &= \lambda_1 Q_\omega(x_1, x_2, \delta_1), & z_2 &= 2^{-j} \lambda_1 \lambda_3^{-1/2},\\
z_3 &= \lambda_1^{-\varepsilon}, & z_4 &= \lambda_3 \lambda_1^{-1}.
\end{aligned}
$$

Note that the parameters λ_3 and x_3 are not bounded.

Applying Lemma 2.2.6 we get

$$
\left\| (\lambda_3)^{-\frac{1}{2} - \frac{3}{4} it} \sum_{\lambda_1} (\lambda_1)^{-1 - \frac{3}{2} it} \nu_j^\lambda \right\|_{L^\infty} \lesssim \frac{|\tilde{H}(0)| + \sum_{k=1}^4 C_k}{|2^{-\frac{3}{2} it} - 1|} \lesssim \frac{\|\tilde{H}\|_{L^\infty} + \sum_{k=1}^4 C_k}{|\gamma(1 + it)|},
$$

if we add an appropriate factor to γ (i.e., our γ needs to contain a factor equal to the expression (2.2.1)). It remains to prove that one can estimate $\|\tilde{H}\|_{L^\infty}$ and the constants C_k, $k = 1, 2, 3, 4$, by $|B|^{-\varepsilon_B}$ since then we can sum in λ_3.

First let us consider the expression for $\tilde{H}(z)$. The first three factors within the integral are behaving essentially like

$$
\check{\chi}_1(y_1) \check{\chi}_1(z_1 - y_1 - z_2 y_2) \check{\chi}_1(B - z_4 y_1 - y_2^2).
$$

Since we could otherwise use the first factor and estimate by $|B|^{-\varepsilon_B}$, we may restrict our discussion to the part of the integration domain where $|y_1| \ll |B|^{\varepsilon_B}$. Then we have $|B - z_4 y_1 r_\beta| \sim |B|$, and therefore

$$
\int |\check{\chi}_1(B - z_4 y_1 r_\beta - y_2^2 b^{\#})| \mathrm{d}y_2 \leqslant 2 \int |\check{\chi}_1(B - z_4 y_1 r_\beta - t)|\, |t|^{-1/2} \mathrm{d}t \leqslant C |B|^{-1/2},
$$

for a constant C. Hence, we have the required bound for $\|\tilde{H}\|_{L^\infty}$.

Next, we see that taking derivatives in z_1 and z_4, doesn't change in an essential way the actual form of $\tilde{H}$ since we only obtain polynomial growth in y_1 which can be absorbed by $\hat{\chi}_1(y_1)$, and since derivatives of Schwartz functions are again Schwartz. Therefore, we may estimate C_k, $k = 1, 4$, in the same way as we estimated the original integral.

Permuting the order of the variables z_k, $k = 1, 2, 3, 4$ appropriately, we see from the expressions for C_k in Lemma 2.2.6 that we may now assume $z_1 = z_4 = 0$. Taking the derivative in z_3 we obtain several terms. We deal with the terms where a y_1 factor appears in the same way as we have dealt with in the previous cases. It remains to deal with the term where y_2^2 factor appears, that is

$$
\begin{aligned}
&- z_3^{-1+1/\varepsilon}\,(\partial_1 b^{\#})(x_1 - z_3^{1/\varepsilon} y_1, \lambda_3^{-1/2} y_2, \delta, j)\\
&\times \int \check{\chi}_1(y_1)\,\check{\chi}_1(z_1 - y_1 r_\omega(z_3^{1/\varepsilon} y_1, x_1, \delta_1) - z_2 y_2)\\
&\times (\check{\chi}_1)'(\lambda_3 Q_\beta(x_1, x_3, \delta_1) - z_4 y_1 r_\beta(z_3^{1/\varepsilon} y_1, x_1, \delta_1) - b^{\#}(x_1 - z_3^{1/\varepsilon} y_1, \lambda_3^{-1/2} y_2, \delta, j) y_2^2)\\
&\times a(x_1 - z_3^{1/\varepsilon} y_1, \lambda_3^{-1/2} y_2, \delta, j)\,\chi_1(x_1 - z_3^{1/\varepsilon} y_1)\,\chi_0(\lambda_3^{-1/2} y_2)\,\chi_0(z_3 y_1)\,y_1 y_2^2\,\mathrm{d}y.
\end{aligned}
$$

This integral can be estimated by

$$
\int |\chi_0(\lambda_3^{-1/2} y_2)\,(\check{\chi}_1)'(B - y_2^2 b^{\#})|\,|z_3|^{-1+1/\varepsilon}\,y_2^2 \mathrm{d}y_2. \tag{3.3.36}
$$

The key is now to notice that if we fix λ_3, then λ_1 goes over the set where $\lambda_1 \gg \lambda_3$. In particular, since we shall plug in $z_3 = \lambda_1^{-\varepsilon}$, we have $|z_3|^{-1+1/\varepsilon} \lesssim \lambda_3^{-1+\varepsilon}$. Therefore using the first factor in (3.3.36) we obtain the bound for (3.3.36) to be

$$
\int |(\check{\chi}_1)'(B - y_2^2 b^{\#})|\,|y_2|^{\varepsilon} \mathrm{d}y_2,
$$

for some different ε. Now one subsitutes $t = y_2^2 b^{\#}$ and easily obtains an admissible bound of the form $|B|^{-\varepsilon_B}$.

For the last constant C_2 we shall need to consider the Hölder norm. Here we may assume $z_1 = z_3 = z_4 = 0$. The derivative in z_2 can be estimated by the integral

$$
\int \Big|\check{\chi}_1(y_1)\,(\check{\chi}_1)'(-y_1 r_\omega(0, x_1, \delta_1) - z_2 y_2)\check{\chi}_1(B - b^{\#}(x_1, \lambda_3^{-1/2} y_2, \delta, j) y_2^2)\,y_2\Big| \mathrm{d}y.
$$

We shall now consider only the part where $y_2 \geqslant 0$ and $z_2 \geqslant 0$, as other cases can be treated in the same way. Then substituting $t = y_2^2$ one gets that the estimate for $\partial_{z_2} \tilde{H}$ is

$$
\iint \Big|\check{\chi}_1(y_1)\,(\check{\chi}_1)'(-y_1 r_\omega - z_2 t^{1/2})\check{\chi}_1(B - t b^{\#})\Big| \mathrm{d}y_1 \mathrm{d}t.
$$

From this form it is obvious that we may now restrict ourselves to the part of the integration domain where $|y_1| \ll |B|^{\varepsilon_B}$ and $|t| \sim |B|$ by using the first and the third factor respectively. If we denote this integration domain by U_B, then the bound for the C_2 constant in Lemma 2.2.6 reduces to estimating

$$
\begin{aligned}
&|z_2|^{1-\vartheta} \int_0^1 \iint_{U_B} \Big|\check{\chi}_1(y_1)\,(\check{\chi}_1)'(-y_1 r_\omega - s z_2 t^{1/2})\check{\chi}_1(B - t b^{\#})\Big| \mathrm{d}y_1 \mathrm{d}t \mathrm{d}s\\
=&|z_2|^{-\vartheta} \int_0^{z_2} \iint_{U_B} \Big|\check{\chi}_1(y_1)\,(\check{\chi}_1)'(-y_1 r_\omega - \tilde{s} t^{1/2})\check{\chi}_1(B - t b^{\#})\Big| \mathrm{d}y_1 \mathrm{d}t \mathrm{d}\tilde{s},
\end{aligned}
$$

where ϑ represents the Hölder exponent. If $|z_2| \leqslant |B|^{-1/4}$, then we obviously have the required estimate. Therefore, let us assume $|z_2| > |B|^{-1/4}$. Then $|z_2|^{-\vartheta} < |B|^{\vartheta/4}$ and so integration on the domain $|\tilde{s}| \leqslant |B|^{-1/4}$ is not a problem. On the other hand, if $|\tilde{s}| > |B|^{-1/4}$, then $|\tilde{s}t^{1/2}| \gtrsim |B|^{1/4}$ by our assumption on the size of t. Thus we may use the Schwartz property of the second factor in the integral and obtain the required estimate. This finishes the proof of the case where $|B| \gg 1$ and $|A| \lesssim 1$.

Subcase $|B| > C_B$ and $|A| \gg 1$. The preceding argumentation for the estimate of $\|\tilde{H}\|_{L^\infty}$ is also valid in this case since we have not used the second factor, and so we see that we can always estimate the integral appearing in (3.3.34) by $|B|^{-1/2}$. It remains to gain a decay in $|A|$.

If we furthermore assume $|A| \leqslant |B|$, then $|B|^{-1/2} \leqslant |B|^{-1/4}|A|^{-1/4}$, and so we can sum in both λ_1 and λ_3. Therefore we may consider $|A| > |B|$ next, and reduce our problem using the first factor in the integral in (3.3.34) to the part where $|y_1| \ll |A|^{\varepsilon_A}$. Then $|z_1 - y_1 r_\omega| = |A - y_1 r_\omega| \sim |A|$, and so we can gain an $|A|^{-\varepsilon_A}$ using the second factor in the integral, unless $|z_2 y_2| \sim |A|$. But since $|z_2| \lesssim 1$, we see that $|z_2 y_2| \sim |A|$ implies $|y_2| \gtrsim |A|$, and so we can use finally the third factor where then the y_2^2 term is dominant.

Subcase $|B| \leqslant C_B$ and $|A| > C_B^2$. We can reduce ourselves to the integration over $|y_1| \ll |A|^{\varepsilon_A}$, and so $|A - y_1 r_\omega| \sim |A|$. Therefore, if $|y_2| \leqslant |A|^{1/2}$, then using the second factor we get that the integral is bounded (up to a constant) by $|A|^{-1}$. If $|y_2| > |A|^{1/2}$, then $|B - z_4 y_1 r_\beta - y_2^2 b^\#| \gtrsim |A|$, and so we can use the third factor, and sum in both λ_1 and λ_3 (since $|B| < |A|$).

Subcase $|B| \leqslant C_B$ and $|A| \leqslant C_B^2$. Finally, if both $|A| \leqslant C_B^2$ and $|B| \leqslant C_B$ are bounded, we use the two parameter oscillatory sum lemma. We define the function

$$
\begin{aligned}
H(z_1, z_2,& z_3, z_4, z_5, z_6; x_1, \delta, 2^{-j}) := \\
&\int \check{\chi}_1(y_1)\, \check{\chi}_1(z_1 - y_1 r_\omega(z_3^{1/\varepsilon} y_1, x_1, \delta_1) - z_5 y_2) \\
&\times \check{\chi}_1(z_2 - z_6 y_1 r_\beta(z_3^{1/\varepsilon} y_1, x_1, \delta_1) - b^\#(x_1 - z_3^{1/\varepsilon} y_1, z_4 y_2, \delta, j) y_2^2) \\
&\times a(x_1 - z_3^{1/\varepsilon} y_1, z_4 y_2, \delta, j)\, \chi_1(x_1 - z_3^{1/\varepsilon} y_1)\, \chi_0(z_4 y_2)\, \chi_0(z_3 y_1)\, \mathrm{d}y,
\end{aligned}
$$

where we shall plug in

$$
\begin{aligned}
z_1 &= \lambda_1 Q_\omega(x_1, x_2, \delta_1), & z_2 &= \lambda_3 Q_\beta(x_1, x_3, \delta_1), \\
z_3 &= \lambda_1^{-\varepsilon}, & z_4 &= \lambda_3^{-1/2}, \\
z_5 &= 2^{-j} \lambda_1 \lambda_3^{-1/2}, & z_6 &= \lambda_3 \lambda_1^{-1}.
\end{aligned}
$$

The associated exponents are $(\alpha_1, \alpha_2) = (-3/2, -3/4)$ and

$$
\begin{aligned}
(\beta_1^1, \beta_2^1) &= (1, 0), & (\beta_1^2, \beta_2^2) &= (0, 1), \\
(\beta_1^3, \beta_2^3) &= (-\varepsilon, 0), & (\beta_1^4, \beta_2^4) &= (0, -1/2), \\
(\beta_1^5, \beta_2^5) &= (1, -1/2), & (\beta_1^6, \beta_2^6) &= (-1, 1),
\end{aligned}
$$

and so for each k the pairs (α_1, α_2) and (β_1^k, β_2^k) are linearly independent. The C^2 norm of H is uniformly bounded in the bounded paramteres $(x_1, \delta, 2^{-j})$ by arguing in the same manner as in Case 1.1. Therefore we may apply Lemma 2.2.7 if we take γ to contain a factor equal to the expression (2.2.3) (with $\theta = 1/3$). Recall that γ needs to contain also the factor from the case where we applied the one parameter lemma (i.e., where we had $|B| > C_B$ and $|A| \lesssim 1$).

Case 3.2. $\lambda_1 \sim \lambda_2$, $\lambda_3 \ll \lambda_1$, and $\lambda_3^{1/2} \ll 2^{-j}\lambda_1$. Here we have the same bound for the Fourier transform as in the previous case. Therefore

$$\|\widehat{\nu_j^\lambda}\|_{L^\infty} \lesssim \lambda_1^{-1/2}\lambda_3^{-1/2}, \qquad \|\nu_j^\lambda\|_{L^\infty} \lesssim 2^j\lambda_3,$$

from which one can get by interpolation

$$\|T_j^\lambda\|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})\to L^{\mathfrak{p}_3'}_{x_3}(L^{\mathfrak{p}_1'}_{(x_1,x_2)})} \lesssim 2^{j\theta}\lambda_1^{(\theta-1)/2}\lambda_3^{\theta/2}.$$

We first consider the case $\lambda_1 > 2^{2j}$ and denote its sum of the operator pieces by $T_{\delta,j}^{VI,1}$. Then

$$\begin{aligned}
\|T_{\delta,j}^{VI,1}\|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})\to L^{\mathfrak{p}_3'}_{x_3}(L^{\mathfrak{p}_1'}_{(x_1,x_2)})} &\lesssim 2^{j\theta}\sum_{\lambda_1=2^{2j}}^{\infty}\sum_{\lambda_3=1}^{\lambda_1}\lambda_1^{(\theta-1)/2}\lambda_3^{\theta/2} \\
&\lesssim 2^{j\theta}\sum_{\lambda_1=2^{2j}}^{\infty}\lambda_1^{\theta-1/2} \\
&\lesssim 2^{j(3\theta-1)} \lesssim 1.
\end{aligned}$$

The other case is when $2^j \ll \lambda_1 \leqslant 2^{2j}$ and we denote the sum of these operator pieces by $T_{\delta,j}^{VI,2}$. Then

$$\begin{aligned}
\|T_{\delta,j}^{VI,2}\|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})\to L^{\mathfrak{p}_3'}_{x_3}(L^{\mathfrak{p}_1'}_{(x_1,x_2)})} &\lesssim 2^{j\theta}\sum_{\lambda_1=2^{j}}^{2^{2j}}\sum_{\lambda_3=1}^{(2^{-j}\lambda_1)^2}\lambda_1^{(\theta-1)/2}\lambda_3^{\theta/2} \\
&\lesssim \sum_{\lambda_1=2^{j}}^{2^{2j}}\lambda_1^{(\theta-1)/2}\lambda_1^{\theta} \\
&\lesssim \sum_{\lambda_1=2^{j}}^{2^{2j}}\lambda_1^{(3\theta-1)/2}.
\end{aligned}$$

Again, this is summable if and only if $\theta < 1/3$. For $\theta = 1/3$, we have

$$\|T_j^\lambda\|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})\to L^{\mathfrak{p}_3'}_{x_3}(L^{\mathfrak{p}_1'}_{(x_1,x_2)})} \lesssim 2^{j/3}(\lambda_1^2\lambda_3^{-1})^{-1/6}.$$

This operator norm estimate motivates us to define k through $2^k := \lambda_1^2\lambda_3^{-1} = 2^{2k_1-k_3}$, where $2^{k_1} = \lambda_1$ and $2^{k_3} = \lambda_3$. Our goal is to prove for each k that

$$\Bigg\|\sum_{\lambda_1^2\lambda_3^{-1}=2^k} T_j^\lambda\Bigg\|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})\to L^{\mathfrak{p}_3'}_{x_3}(L^{\mathfrak{p}_1'}_{(x_1,x_2)})} \lesssim 2^{(2\varepsilon-1)(k-2j)/3}$$

for some $0 \leqslant \varepsilon < 1/2$. Since $k \geqslant 2j$, we then obtain the desired result by summation in k.

We shall slightly simplify the proof by assuming that $\lambda_1 = \lambda_2$. Let us consider the following function parametrized by the complex number ζ and k:

$$\mu_\zeta^k = \gamma(\zeta) \sum_{\lambda_1^2\lambda_3^{-1}=2^k} (\lambda_3)^{\frac{1-3\zeta}{2}} \nu_j^\lambda,$$

where

$$\gamma(\zeta) = 2^{-3(\zeta-1)/2} - 1.$$

Let T_ζ^k denote the associated convolution operator. For $\zeta = 1/3$ we have

$$\mu_\zeta^k = \sum_{\lambda_1^2\lambda_3^{-1}=2^k} \nu_j^\lambda,$$

and so, by interpolation, we need to prove

$$\begin{aligned} \|T_{it}^k\|_{L_{x_3}^{2/(2-\tilde{\sigma})}(L^1_{(x_1,x_2)}) \to L_{x_3}^{2/\tilde{\sigma}}(L^\infty_{(x_1,x_2)})} &\lesssim 2^{-k/4}, \\ \|T_{1+it}^k\|_{L^2\to L^2} &\lesssim 2^j\, 2^{\varepsilon(k-2j)}, \end{aligned} \tag{3.3.37}$$

for some $0 \leqslant \varepsilon < 1/2$, and with constants uniform in $t \in \mathbb{R}$. The first estimate follows right away since $\widehat{\nu_j^\lambda}$ have supports located at λ, and therefore by the L^∞ estimate for the Fourier transform of ν_j^λ we have

$$|\widehat{\mu_{it}^k}(\xi)| \lesssim \frac{2^{-k/4}}{(1+|\xi_3|)^{1/4}}.$$

We prove the second estimate by using the oscillatory sum lemma. We need to prove

$$\begin{aligned} &\Bigg\| \sum_{\lambda_1^2\lambda_3^{-1}=2^k} \lambda_3^{-1-\frac{3}{2}it} \nu_j^\lambda \Bigg\|_{L^\infty} = \\ &\qquad \Bigg\| \sum_{\lambda_1^2\lambda_3^{-1}=2^k} 2^k \lambda_1^{-2-3it} \nu_j^{(\lambda_1,\lambda_1,2^{-k}\lambda_1^2)} \Bigg\|_{L^\infty} \lesssim \frac{2^j\, 2^{\varepsilon(k-2j)}}{\left|2^{-\frac{3}{2}it} - 1\right|}, \end{aligned} \tag{3.3.38}$$

uniformly in t.

Let us discuss first the index ranges for λ_1, λ_3, and $2^k = \lambda_1^2\lambda_3^{-1}$. Recall that we are in the case where $2^j \ll \lambda_1 \leqslant 2^{2j}$ and $1 \leqslant \lambda_3 \ll \lambda_1^2 2^{-2j}$, which implies $\lambda_3 \ll \lambda_1$ and $2^{2j} \ll 2^k \leqslant 2^{4j}$. Let us now fix any k satisfying $2^{2j} \ll 2^k \leqslant 2^{4j}$, and let us consider all (λ_1, λ_3) such that $2^k = \lambda_1^2\lambda_3^{-1}$. We shall use the oscillatory sum lemma by summing in λ_1 and consider $\lambda_3 = \lambda_1^2 2^{-k}$ as a function of λ_1 and k. The conditions for λ_1 are then

$$\begin{aligned} 2^j \ll \lambda_1 &\leqslant 2^{2j}, \\ 1 \leqslant \lambda_1^2 2^{-k} &\ll 2^{2j}, \end{aligned}$$

which determine an interval of integers $I_{j,k}$ for k_1 (recall $\lambda_1 = 2^{k_1}$).

After substituting $\lambda_1 y_1 \mapsto y_1$ and $2^{-j}\lambda_1 y_2 \mapsto y_2$ in the expression (3.3.27), we get that the sum on the left hand side of (3.3.38) is

$$\begin{aligned} 2^j \sum_{k_1 \in I_{j,k}} \lambda_1^{-3it} \int & \check{\chi}_1(\lambda_1 x_1 - y_1)\, \check{\chi}_1(\lambda_1 x_2 - y_2 - \lambda_1 \psi_\omega(\lambda_1^{-1} y_1)) \\ & \times \check{\chi}_1(\lambda_3 x_3 - 2^{2j-k} b^\#(\lambda_1^{-1} y_1, 2^j \lambda_1^{-1} y_2, \delta, j) y_2^2 - \lambda_3 \psi_\beta(\lambda_1^{-1} y_1)) \\ & \times a(\lambda_1^{-1} y_1, 2^j \lambda_1^{-1} y_2, \delta, j)\, \chi_1(\lambda_1^{-1} y_1)\, \chi_0(2^j \lambda_1^{-1} y_2)\, \mathrm{d}y. \end{aligned}$$

Since using the first two factors we can get a decay in λ_1, we can restrict ourselves to the case $|(x_1, x_2)| \lesssim 1$. When $|x_3| \gg 1$, then by using the third factor we can gain a factor $\lambda_3^{-1} = (\lambda_1^2 2^{-k})^{-1}$, which sums up to a number of size ~ 1, by definition of $I_{j,k}$. Therefore we may and shall assume $|x| \lesssim 1$.

Next, we use the substitution $\lambda_1 x_1 - y_1 \mapsto y_1$, conclude that it is sufficient to consider the part of the integration domain where $|y_1| \leqslant \lambda_1^\varepsilon$, and that we may assume $x_1 \sim 1$. If we use Taylor approximation for ψ_ω and ψ_β at x_1, then one gets

$$\begin{aligned} 2^j \sum_{k_1 \in I_{j,k}} \lambda_1^{-3it} \int & \check{\chi}_1(y_1)\, \check{\chi}_1(\lambda_1 Q_\omega(x_1, x_2, \delta_1) - y_1 r_\omega(\lambda_1^{-1} y_1, x_1, \delta_1) - y_2) \\ & \times \check{\chi}_1(\lambda_3 Q_\beta(x_1, x_3, \delta_1) - \lambda_3 \lambda_1^{-1} y_1 r_\beta(\lambda_1^{-1} y_1, x_1, \delta_1) \\ & \qquad\qquad - 2^{2j-k} b^\#(x_1 - \lambda_1^{-1} y_1, 2^j \lambda_1^{-1} y_2, \delta, j) y_2^2) \\ & \times a(x_1 - \lambda_1^{-1} y_1, 2^j \lambda_1^{-1} y_2, \delta, j)\, \chi_1(x_1 - \lambda_1^{-1} y_1)\, \chi_0(2^j \lambda_1^{-1} y_2)\, \chi_0(\lambda_1^{-\varepsilon} y_1)\, \mathrm{d}y, \end{aligned}$$

where $|\partial_1^N r_\omega| \sim 1$ and $|\partial_1^N r_\beta| \sim 1$ for any $N \geqslant 0$. Note that $2^{2j-k} \ll 1$ and $\lambda_3 \lambda_1^{-1} \ll 1$, and therefore it is sufficient to consider the cases when either $|A| \gg 1$ or $|B| \gg 1$, where

$$A := \lambda_1 Q_\omega(x_1, x_2, \delta_1), \qquad B := \lambda_3 Q_\beta(x_1, x_3, \delta_1),$$

since otherwise we may use the oscillatory sum lemma. We remind that $\lambda_3 = \lambda_1^2 2^{-k}$ is considered to be a function of λ_1.

We concentrate on the first three factors within the integral:

$$\check{\chi}_1(y_1) \check{\chi}_1(A - r_\omega y_1 - y_2) \check{\chi}_1(B - \lambda_3 \lambda_1^{-1} r_\beta y_1 - 2^{2j-k} b^\# y_2^2),$$

where r_ω, r_β, and $b^\#$ are all converging in C^∞ to constant functions of magnitude ~ 1 when $\lambda_1 \to \infty$, $\delta \to 0$, and $j \to \infty$.

Let us denote by M a large enough positive number.

Subcase $|B| > M^3$ and $|A| \leqslant M$. Then because of the first factor we may restrict our discussion to the integration domain where $|y_1| < |B|^{1/3}$. There $|A - r_\omega y_1| \leqslant C|B|^{1/3}$ for some C. We may then furthermore assume $|y_2| \leqslant 2C|B|^{1/3}$, since otherwise we could use the second factor. Now, if we take M sufficiently large, we have

$$|\lambda_3 \lambda_1^{-1} r_\beta y_1 - 2^{2j-k} b^\# y_2^2| \ll B,$$

and so we can now use the third factor's Schwartz property to obtain a factor $|B|^{-1}$, which gives summability in λ_1.

Subcase $|A| > M$. Here we shall need a slightly finer analysis. Note that using the first factor within the integral we can actually reduce ourselves to the integration within the slightly narrower range $|y_1| < |A|^{\varepsilon} 2^{10\varepsilon(2j-k)}$ for some small ε (see (3.3.37)), and therefore we can also assume using the second factor that

$$y_2 \in [A - C|A|^{\varepsilon} 2^{10\varepsilon(2j-k)}, A + C|A|^{\varepsilon} 2^{10\varepsilon(2j-k)}],$$

for some C.

Now if $|A|^{\varepsilon} 2^{10\varepsilon(2j-k)} \leqslant 1$, we obtain that the bound on the integral is $|A|^{2\varepsilon} 2^{20\varepsilon(2j-k)}$ (the area of the surface over which we integrate), and this is summable in λ_1 over the set $|A|^{\varepsilon} 2^{10\varepsilon(2j-k)} \leqslant 1$.

Therefore, we assume $|A|^{\varepsilon} 2^{10\varepsilon(2j-k)} > 1$, that is $|A|^{1/10} > 2^{k-2j}$. Now, if M is sufficiently large, we then have by the restraint on y_2 that $|A|^2/2 < y_2^2 < 2|A|^2$, and hence

$$C_1 |A|^{2-1/10} < |2^{2j-k} b^{\#} y_2^2| < C_2 |A|^2.$$

Therefore if either $|B| \ll C_1|A|^{2-1/10}$ or $|B| \gg C_2|A|^2$, we can simply use the Schwartz property of the third factor within the integral. Let us now assume that B is within the range $|B| \in [C_1|A|^{2-1/10}, C_2|A|^2]$. We denote $\delta_A := |A|^{\varepsilon} 2^{10\varepsilon(2j-k)}$ and recall $\delta_A > 1$ and $|y_1| < \delta_A \leqslant |A|^{\varepsilon}$. Using the third factor within the integral we can reduce our problem to when

$$|B - \lambda_3 \lambda_1^{-1} r_{\beta} y_1 - 2^{2j-k} b^{\#} y_2^2| \leqslant \delta_A.$$

The implicit function theorem implies that

$$\begin{aligned} & y_2^2 \in [2^{k-2j}|B| - C' 2^{k-2j} \delta_A, 2^{k-2j}|B| + C' 2^{k-2j} \delta_A] \\ \iff & \frac{y_2^2}{2^{k-2j}|B|} \in \Big[1 - \frac{C'\delta_A}{|B|}, 1 + \frac{C\delta_A}{|B|}\Big], \end{aligned}$$

for some C'. Since $\delta_A \leqslant |A|^{1/10}$ and $|B| > |A|^{3/2}$, we can conclude

$$|y_2| \in [(2^{k-2j}|B|)^{1/2} - |A|^{-1/2}, (2^{k-2j}|B|)^{1/2} + |A|^{-1/2}],$$

that is, y_2 goes over a set with length at most $C'|A|^{-1/2}$. This implies that our integral is bounded by $C'|A|^{-1/2+\varepsilon}$, which is summable in λ_1.

Case 4.1. $\lambda_1 \sim \lambda_2 \sim \lambda_3$ and $\lambda_1 > 2^{2j}$. Here one first applies stationary phase in x_2. Afterwards, as easily seen and explained in a bit more detail at the end of [51, Chapter 4] (and also in the following Section 3.4), one gets a phase function in x_1 which has a singularity of Airy-type. By using Lemma 2.2.1, with condition (ii) and $M = 3$, one gets that the Fourier transform estimate is

$$\|\widehat{\nu_j^{\lambda}}\|_{L^\infty} \lesssim \lambda_1^{-1/2} \lambda_1^{-1/3} = \lambda_1^{-5/6}.$$

From (3.3.28) the space-side estimate is

$$\|\nu_j^{\lambda}\|_{L^\infty} \lesssim 2^j \lambda_1,$$

from which one gets by interpolation

$$\|T_j^\lambda\|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})\to L^{\mathfrak{p}_3'}_{x_3}(L^{\mathfrak{p}_1'}_{(x_1,x_2)})} \lesssim 2^{j\theta}\lambda_1^{(5\theta-2)/6}.$$

The bound on the operator norm is

$$\begin{aligned}\|\tilde{T}^{VII}_{\delta,j}\|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})\to L^{\mathfrak{p}_3'}_{x_3}(L^{\mathfrak{p}_1'}_{(x_1,x_2)})} &\lesssim 2^{j\theta}\sum_{\lambda_1=2^{2j}}^{\infty}\lambda_1^{(5\theta-2)/6}\\ &\lesssim 2^{j(8\theta-2)/3},\end{aligned}$$

where $\tilde{T}^{VII}_{\delta,j}$ denotes the sum of the associated operator pieces. This is uniformly bounded if and only if $\theta \leqslant 1/4$. For $\theta = 1/3$, we can only sum in the range $\lambda_1 > 2^{6j}$ and so it remains to see what happens when $2^{2j} < \lambda_1 \leqslant 2^{6j}$. We denote the sum of the associated operator pieces for this remaining range by $T^{VII}_{\delta,j}$. We shall deal with this case in the following section.

Case 4.2. $\lambda_1 \sim \lambda_2 \sim \lambda_3$ **and** $\lambda_1 \leqslant 2^{2j}$**.** Here only the space-side estimate changes and we have

$$\|\widehat{\nu_j^\lambda}\|_{L^\infty} \lesssim \lambda_1^{-5/6}, \qquad \|\nu_j^\lambda\|_{L^\infty} \lesssim \lambda_1^{3/2}. \tag{3.3.39}$$

By interpolation one can obtain

$$\|T_j^\lambda\|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})\to L^{\mathfrak{p}_3'}_{x_3}(L^{\mathfrak{p}_1'}_{(x_1,x_2)})} \lesssim \lambda_1^{(4\theta-1)/3}.$$

We denote the sum of the associated operator pieces by $T^{VIII}_{\delta,j}$. The above estimate is obviously summable if and only if $\theta < 1/4$. For $\theta = 1/4$ we shall now use complex interpolation, and we deal with $\theta = 1/3$ in the next section. We obviously may assume in this case $\lambda_i \gg 1$ for all $i = 1, 2, 3$.

For simplicity, we assume that $\lambda_1 = \lambda_2 = \lambda_3$. We consider the following function parametrized by the complex number ζ:

$$\mu_\zeta = \gamma(\zeta)\sum_{1\ll\lambda_1\leqslant 2^{2j}}(\lambda_1)^{\frac{1-4\zeta}{2}}\,\nu_j^\lambda, \tag{3.3.40}$$

where

$$\gamma(\zeta) = \frac{2^{-2(\zeta-1)}-1}{2^{\frac{3}{2}}-1}.$$

The associated operator is denoted by T_ζ. For $\zeta = 1/4$ it holds

$$\mu_\zeta = \sum_{1\ll\lambda_1\leqslant 2^{2j}}\nu_j^\lambda,$$

and so by Stein's interpolation theorem it suffices to prove

$$\begin{aligned}\|T_{it}\|_{L^{2/(2-\tilde{\sigma})}_{x_3}(L^1_{(x_1,x_2)})\to L^{2/\tilde{\sigma}}_{x_3}(L^\infty_{(x_1,x_2)})} &\lesssim 1,\\ \|T_{1+it}\|_{L^2\to L^2} &\lesssim 1,\end{aligned}$$

with constants uniform in $t \in \mathbb{R}$. Here $\tilde{\sigma} = 1/3$ since $\theta = 1/4$.

In order to prove the first estimate, we need the decay bound (2.3.2), i.e.,

$$|\widehat{\mu_{it}}(\xi)| \lesssim \frac{1}{(1+|\xi_3|)^{1/3}}.$$

But this follows automatically by (3.3.39), the definition of μ_ζ, and the fact that each $\widehat{\nu_j^\lambda}$ has its support located around λ.

We prove the second $L^2 \to L^2$ estimate by using the oscillatory sum lemma [51, Lemma 2.7]. We need to prove

$$\Big\| \sum_{1 \ll \lambda_1 \leqslant 2^{2j}} (\lambda_1)^{-\frac{3}{2}-2it}\, \nu_j^\lambda \Big\|_{L^\infty} \lesssim \frac{1}{\left|2^{-2it}-1\right|}, \tag{3.3.41}$$

uniformly in t.

After substituting $\lambda_1 y_1 \mapsto y_1$ and $\lambda_1^{1/2} y_2 \mapsto y_2$ in the expression (3.3.27), we get that the sum on the left hand side of (3.3.41) is

$$\begin{aligned}\sum_{1 \ll \lambda_1 \leqslant 2^{2j}} \lambda_1^{-2it} \int & \check{\chi}_1(\lambda_1 x_1 - y_1)\, \check{\chi}_1(\lambda_1 x_2 - 2^{-j}\lambda_1^{1/2} y_2 - \lambda_1 \psi_\omega(\lambda_1^{-1} y_1)) \\ & \times \check{\chi}_1(\lambda_1 x_3 - b^\#(\lambda_1^{-1} y_1, \lambda_1^{-1/2} y_2, \delta, j) y_2^2 - \lambda_1 \psi_\beta(\lambda_1^{-1} y_1)) \\ & \times a(\lambda_1^{-1} y_1, \lambda_1^{-1/2} y_2, \delta, j)\, \chi_1(\lambda_1^{-1} y_1)\, \chi_0(\lambda_1^{-1/2} y_2)\, \mathrm{d}y.\end{aligned}$$

We may assume that $|x| \leqslant C$ for some large constant C, since otherwise we could use the first three factors to gain a decay in λ_1.

Now as usual, we use the substitution $\lambda_1 x_1 - y_1 \mapsto y_1$, conclude that it is sufficient to consider $|y_1| \leqslant \lambda_1^\varepsilon$ and $x_1 \sim 1$, and use Taylor approximation for ψ_ω and ψ_β at x_1. Then one gets

$$\begin{aligned}\sum_{1 \ll \lambda_1 \leqslant 2^{2j}} \lambda_1^{-2it} \int & \check{\chi}_1(y_1)\, \check{\chi}_1(\lambda_1 Q_\omega(x_1, x_2, \delta_1) - y_1 r_\omega(\lambda_1^{-1} y_1, x_1, \delta_1) - 2^{-j}\lambda_1^{1/2} y_2) \\ & \times \check{\chi}_1(\lambda_1 Q_\beta(x_1, x_3, \delta_1) - y_1 r_\beta(\lambda_1^{-1} y_1, x_1, \delta_1) - b^\#(x_1 - \lambda_1^{-1} y_1, \lambda_1^{-1/2} y_2, \delta, j) y_2^2) \\ & \times a(x_1 - \lambda_1^{-1} y_1, \lambda_1^{-1/2} y_2, \delta, j)\, \chi_1(x_1 - \lambda_1^{-1} y_1)\, \chi_0(\lambda_1^{-1/2} y_2)\, \chi_0(\lambda_1^{-\varepsilon} y_1)\, \mathrm{d}y,\end{aligned}$$

where $|\partial_1^N r_\omega| \sim 1$ and $|\partial_1^N r_\beta| \sim 1$ for any $N \geqslant 0$. Note that $2^{-j}\lambda_1^{1/2} \leqslant 1$, and therefore it is sufficient to consider the cases when either $|A| \gg 1$ or $|B| \gg 1$, where

$$A := \lambda_1 Q_\omega(x_1, x_2, \delta_1), \qquad B := \lambda_1 Q_\beta(x_1, x_3, \delta_1),$$

since otherwise we can use the oscillatory sum lemma.

The first three factors within the integral are behaving essentially like

$$\check{\chi}_1(y_1) \check{\chi}_1(A - y_1 - 2^{-j}\lambda_1^{1/2} y_2) \check{\chi}_1(B - y_1 - y_2^2).$$

If $|B| \gg 1$, then since we could otherwise use the first factor, we can assume $|y_1| \ll |B|^{\varepsilon_B}$. Then $|B - y_1 r_\beta| \sim |B|$, and we can estimate the integral by

$$\int |\check{\chi}_1(B - y_1 r_\beta - y_2^2 b^\#)| \mathrm{d}y_2 \lesssim \int |\check{\chi}_1(B - y_1 r_\beta - t)| \, |t|^{-1/2} \mathrm{d}t \lesssim |B|^{-1/2}.$$

Now one can sum in λ_1.

Let us now assume $|B| \leqslant C_B$ for some large, but fixed constant C_B, and $|A| \gg C_B$. Again, we can assume $|y_1| \ll |A|^{\varepsilon_A}$, and so $|A - y_1 r_\omega| \sim |A|$. Therefore if $|y_2| \leqslant |A|^{1/2}$, then using the second factor we get that the integral is bounded (up to a constant) by $|A|^{-N}$. If $|y_2| > |A|^{1/2}$, then $|B - y_1 r_\beta - y_2^2 b^\#| \gtrsim |A|$ and so we can use the third factor, and sum in λ_1.

3.4 Airy-type analysis in the case $h_{\mathrm{lin}}(\phi) < 2$

In this section we begin with the proof of the estimates for $T_{\delta,j}^{VII}$ and $T_{\delta,j}^{VIII}$ when $\theta = 1/3$, i.e., when ϕ is of type A_{n-1} with $m = 2$ and finite $n \geqslant 5$. In this case $\tilde{\sigma} = 1/4$. We shall first recall some of the notation from [51, Chapter 5]. From now on δ shall be a triple $(\delta_0, \delta_1, \delta_2)$ with $\delta_0 = 2^{-j}$, we use λ to denote the common value $\lambda_1 = \lambda_2 = \lambda_3$, and define

$$\begin{aligned} s_1 &:= \frac{\xi_1}{\xi_3}, \quad s_2 := \frac{\xi_2}{\xi_3}, \quad s_3 := \frac{\xi_3}{\lambda}, \\ s &:= (s_1, s_2, s_3), \quad s' := (s_1, s_2). \end{aligned}$$

Then $|s_i| \sim 1$ for $i = 1, 2, 3$, and we have

$$\begin{aligned} \xi &= \lambda s_3 (s_1, s_2, 1), \\ \Phi(x, \delta, j, \xi) &= \lambda s_3 \tilde{\Phi}(x, \delta, \sigma, s_1, s_2), \end{aligned}$$

where Φ is the total phase from (3.3.26) and

$$\begin{aligned} \tilde{\Phi}(x, \delta, j, s_1, s_2) = s_1 x_1 &+ s_2 x_1^2 \omega(\delta_1 x_1) + \sigma x_1^n \beta(\delta_1 x_1) \\ &+ \delta_0 s_2 x_2 + x_2^2 b_0(x, \delta). \end{aligned}$$

Recall that according to Case 4.1 and Case 4.2 from the last subsection of the previous section we have

$$\|T_j^\lambda\|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)}) \to L^{\mathfrak{p}_3'}_{x_3}(L^{\mathfrak{p}_1'}_{(x_1,x_2)})} \lesssim \lambda^{1/9},$$

and we can assume $\lambda \gg 1$. Furthermore, recall that $\sigma \sim 1$, and that

$$b^\#(y, \delta_1, \delta_2, j) = b_0(y, \delta) := b^a(\delta_1 y_1, \delta_0 \delta_2 y_2),$$

where b^a is the same function as in Subsection 3.1.4. It is the function b from Subsection 3.1.4 expressed in adapted coordinates. Recall that $\beta(0) \neq 0$, $\omega(0) \neq 0$, and $b_0(y, 0) = b^a(0, 0) = b(0, 0) \neq 0$ for all y.

In terms of s the expression for the Fourier transform of $\nu_\delta^\lambda := \nu_j^\lambda$ becomes

$$\chi_1(s_1 s_3)\chi_1(s_2 s_3)\chi_1(s_3)\int e^{-i\lambda s_3 \tilde{\Phi}(y,\delta,\sigma,s_1,s_2)}\tilde{a}(y,\delta)\mathrm{d}y,$$

where the amplitude $\tilde{a}(y,\delta) := a(y,\delta)\chi_1(y_1)\chi_0(y_2)$ is a smooth function supported in the sets where $x_1 \sim 1$ and $|x_2| \lesssim 1$ and whose derivatives are uniformly bounded with respect to δ. If we denote

$$T_\delta^\lambda f := f * \widehat{\nu_\delta^\lambda},$$

then the estimate we need to prove is

$$\Big\| \sum_{1 \ll \lambda \leqslant \delta_0^{-6}} T_\delta^\lambda \Big\|_{L_{x_3}^{\mathfrak{p}_3}(L_{(x_1,x_2)}^{\mathfrak{p}_1}) \to L_{x_3}^{\mathfrak{p}_3'}(L_{(x_1,x_2)}^{\mathfrak{p}_1'})} \lesssim 1$$

for

$$\Big(\frac{1}{\mathfrak{p}_1'}, \frac{1}{\mathfrak{p}_3'}\Big) = \Big(\frac{1}{6}, \frac{1}{4}\Big).$$

This estimate corresponds to the estimate of the sum $T_{\delta,j}^{VII} + T_{\delta,j}^{VIII}$ considered in the last subsection of the previous section (Case 4.1 and Case 4.2).

3.4.1 First steps and estimates

Our first step is to use the stationary phase in the y_2 variable, ignoring the part away from the critical point where we can obtain absolutely summable estimates. Then, as explained in [51, Section 5.1], one obtains by using the implicit function theorem that the critical point x_2^c can be written as

$$x_2^c = \delta_0 s_2 Y_2(\delta_1 x_1, \delta_2, \delta_0 s_2),$$

where Y_2 is smooth, $Y_2(0,0,0) = -1/(2b(0,0))$, and $|Y_2| \sim 1$. Now one defines

$$\Psi(x_1,\delta,\sigma,s') := \tilde{\Phi}(x_1, x_2^c, \sigma, s'),$$

so we can write

$$\widehat{\nu_\delta^\lambda}(\xi) = \lambda^{-1/2}\chi_1(s_1 s_3)\chi_1(s_2 s_3)\chi_1(s_3)\int e^{-i\lambda s_3 \Psi(y_1,\delta,\sigma,s_1,s_2)} a_0(y_1,s,\delta;\lambda)\mathrm{d}y_1,$$

where a_0 is smooth and uniformly a classical symbol of order 0 with respect to λ, and where

$$\Psi(y_1,\delta,\sigma,s_1,s_2) = s_1 y_1 + s_2 y_1^2 \omega(\delta_1 y_1) + \sigma y_1^n \beta(\delta_1 y_1) + (\delta_0 s_2)^2 Y_3(\delta_1 y_1, \delta_2, \delta_0 s_2) \quad (3.4.1)$$

for a smooth Y_3 with $Y_3(0,0,0) = -1/4b(0,0) \neq 0$.

Recall that as a_0 is a classical symbol we can express it as

$$a_0(y_1,s,\delta;\lambda) = a_0^0(y_1,s,\delta) + \lambda^{-1}a_0^1(y_1,s,\delta;\lambda),$$

where a_0^0 does not depend on λ and a_0^1 has the same properties as a_0. This induces the decomposition

$$\nu_\delta^\lambda = \nu_{\delta,a_0^0}^\lambda + \nu_{\delta,a_0^1}^\lambda.$$

The function $\nu_{\delta,a_0^1}^\lambda$ associated to the amplitude a_0^1 has Fourier transform bounded by $\lambda^{-3/2}$ and the L^∞ norm on the space side is bounded by $\lambda^{3/2}$ (by the same reasoning as used to obtain (3.3.39)). From these two bounds we can easily get the required estimate for the operator associated to $\nu_{\delta,a_0^1}^\lambda$. Therefore from now on, by an abuse of notation, we may and shall assume that ν_δ^λ has an amplitude which does not depend on λ, i.e.,

$$\widehat{\nu_\delta^\lambda}(\xi) = \lambda^{-1/2}\chi_1(s_1 s_3)\chi_1(s_2 s_3)\chi_1(s_3)\int e^{-i\lambda s_3 \Psi(y_1,\delta,\sigma,s_1,s_2)} a_0(y_1,s,\delta)\mathrm{d}y_1.$$

The next step is to localize the integration in the above integral to a small neighbourhood of the point where the second derivative vanishes. For $\delta = 0$ this point is

$$x_1^c = x_1^c(0,\sigma,s_2) := \left(-\frac{2\omega(0)}{n(n-1)\sigma\beta(0)} s_2\right)^{1/(n-2)}.$$

Away from this point the estimate for the integral is at worst λ^{-1}, by stationary phase or integration by parts.

We now briefly explain how to deal with the part away from x_1^c. Recall from Case 4 in the last subsection of the previous section that the space bound on ν_δ^λ is $2^j\lambda = \delta_0^{-1}\lambda$ if $\lambda > 2^{2j} = \delta_0^{-2}$. Now using the results from Section 2.3 one can easily see that we can sum absolutely in $\lambda > \delta_0^{-2}$. The case when $\lambda \leqslant \delta_0^{-2}$ has to be dealt with complex interpolation as in the Case 4.2 from the last subsection of the previous section. In fact, the proof is completely the same, except that one needs to appropriately change γ and the exponent over λ_1 in the expression for μ_ζ in (3.3.40) since $\theta = 1/3$ in this case, and there it was $\theta = 1/4$. One also has a different amplitude a localising near x_2^c in y_2 integration and away from x_1^c in y_1 integration.

Hence we may now consider only the part near the critical point x_1^c. Abusing the notation again, we shall denote the part near the critical point x_1^c by ν_δ^λ too. Following [51] we shall furthermore assume without loss of generality

$$-\frac{2\omega(0)}{n(n-1)\sigma\beta(0)} = 1, \qquad s_2 \sim 1, \tag{3.4.2}$$

and that in (3.4.1) we are integrating over an arbitrarily small neighbourhood of x_1^c. Therefore, we now have $x_1^c(0,\sigma,s_2) = s_2^{1/(n-2)}$, $|\Psi'''(x_1^c(\delta,\sigma,s_2),\delta,\sigma,s_1,s_2)| \sim 1$, (by implicit function theorem) $x_1^c = x_1^c(\delta,\sigma,s_2)$ depends smoothly in all of its variables, and

$$\Psi''(x_1^c(\delta,\sigma,s_2),\delta,\sigma,s_1,s_2) = 0.$$

We restate [51, Lemma 5.2.] how to locally develop Ψ at the critical point of Ψ', i.e., the point x_1^c. Its proof is straightforward.

Lemma 3.4.1. *The phase* Ψ *given by* (3.4.1) *can be developed locally around* x_1^c *in the form*

$$\Psi(x_1^c(\delta,\sigma,s_2)+y_1,\delta,\sigma,s_1,s_2) = B_0(s',\delta,\sigma) - B_1(s',\delta,\sigma)y_1 + B_3(s_2,\delta,\sigma,y_1)y_1^3,$$

where B_0, B_1, *and* B_3 *are smooth functions, and where* $|B_3(s_2,\delta,\sigma,y_1)| \sim 1$. *In fact, we can write (after taking* (3.4.2) *into account)*

$$\begin{aligned}
x_1^c(\delta,\sigma,s_2) &= s_2^{1/(n-2)}G_1(s_2,\delta,\sigma),\\
B_0(s',\delta,\sigma) &= s_1 s_2^{1/(n-2)}G_1(s_2,\delta,\sigma) - s_2^{n/(n-2)}G_2(s_2,\delta,\sigma),\\
B_1(s',\delta,\sigma) &= -s_1 + s_2^{(n-1)/(n-2)}G_3(s_2,\delta,\sigma),\\
B_3(s',\delta,\sigma,0) &= s_2^{(n-3)/(n-2)}G_4(s_2,\delta,\sigma),
\end{aligned}$$

where G_k, $k = 1,2,3,4$, *are all smooth and of the following forms at* $\delta = 0$*:*

$$\begin{aligned}
G_1(s_2,0,\sigma) &= 1,\\
G_2(s_2,0,\sigma) &= \frac{n^2-n-2}{2}\sigma\beta(0),\\
G_3(s_2,0,\sigma) &= n(n-2)\sigma\beta(0),\\
G_4(s_2,0,\sigma) &= \frac{n(n-1)(n-2)}{6}\sigma\beta(0).
\end{aligned} \tag{3.4.3}$$

We shall also need $G_5 := G_1G_3 - G_2$. *One can easily check that* $G_k \neq 0$ *for each* $k = 1,2,3,4$, *since* $n \geqslant 5$, *and that*

$$G_5(s_2,0,\sigma) = \frac{(n-1)(n-2)}{2}\sigma\beta(0) \neq 0.$$

By applying the lemma we may now write

$$\begin{aligned}
\widehat{\nu_\delta^\lambda}(\xi) =& \lambda^{-1/2}\chi_1(s_1s_3)\chi_1(s_2s_3)\chi_1(s_3)e^{-i\lambda s_3 B_0(s',\delta,\sigma)}\\
&\int e^{-i\lambda s_3(B_3(s_2,\delta,\sigma,y_1)y_1^3 - B_1(s',\delta,\sigma)y_1)}a_0(y_1,s,\delta)\chi_0(y_1)\,\mathrm{d}y_1,
\end{aligned} \tag{3.4.4}$$

where χ_0 is supported here in a sufficiently small neighbourhood of the origin and a_0 denotes a slightly different function than before, but with the same relevant properties. We now decompose $\widehat{\nu_\delta^\lambda}$ further, motivated by Lemma 2.2.2, into parts where $\lambda^{2/3}|B_1(s',\delta,\sigma)| \lesssim 1$ near the Airy cone, and $(2^{-l}\lambda)^{2/3}|B_1(s,\delta,\sigma)| \sim 1$ away from the Airy cone, for $M_0 \leqslant 2^l \leqslant \lambda/M_1$, where M_0, M_1 are sufficiently large. The Airy cone itself is given by the equation $B_1 = 0$.

In order to obtain such a decomposition we take smooth cutoff functions χ_0 and χ_1 such that χ_0 is supported in a sufficiently large neighbourhood of the origin and $\chi_1(t)$ is supported in a neighbourhood of the points -1 and 1 and away from the origin. We furthermore assume that

$$\sum_{l\in\mathbb{Z}} \chi_1(2^{-2l/3}t) = 1$$

on $\mathbb{R}\backslash\{0\}$. Then we can define

$$\begin{aligned}\widehat{\nu_{\delta,Ai}^{\lambda}}(\xi) &:= \chi_0(\lambda^{2/3}B_1(s',\delta,\sigma))\widehat{\nu_{\delta}^{\lambda}}(\xi),\\ \widehat{\nu_{\delta,l}^{\lambda}}(\xi) &:= \chi_1((2^{-l}\lambda)^{2/3}B_1(s',\delta,\sigma))\widehat{\nu_{\delta}^{\lambda}}(\xi),\end{aligned} \tag{3.4.5}$$

where $M_0 \leqslant 2^l \leqslant \lambda/M_1$, so that

$$\nu_{\delta}^{\lambda}(\xi) = \nu_{\delta,Ai}^{\lambda} + \sum_{M_0 \leqslant 2^l \leqslant \lambda/M_1} \nu_{\delta,l}^{\lambda}.$$

We denote the associated convolution operators, convolving against the Fourier transform of $\nu_{\delta,Ai}^{\lambda}$ and $\nu_{\delta,l}^{\lambda}$, by $T_{\delta,Ai}^{\lambda}$ and $T_{\delta,l}^{\lambda}$. Note that the size of the number M_0 is related to how large of a neighbourhood of 0 the cutoff function χ_0 covers in the first equation of (3.4.5), and the size of the number M_1 is related to how small of a neighbourhood of 0 we take in (3.4.4) for the y_1 variable.

3.4.2 Estimates near the Airy cone

From Lemma 2.2.2, (a), we get that the bound on the Fourier transform of $\nu_{\delta,Ai}^{\lambda}$ is $\lambda^{-5/6}$. Unlike in [51] we shall need to use complex interpolation to be able to estimate the part $T_{\delta,Ai}^{\lambda}$. The proof here is actually similar to certain cases when $h_{\text{lin}}(\phi) \geqslant 2$ in [51, Subsection 8.7.1].

We consider the following function parametrized by $\zeta \in \mathbb{C}$:

$$\mu_\zeta = \gamma(\zeta) \sum_{1 \ll \lambda \leqslant \delta_0^{-6}} \lambda^{\frac{7-21\zeta}{12}} \nu_{\delta,Ai}^{\lambda},$$

where

$$\gamma(\zeta) = \frac{2^{-\frac{7(\zeta-1)}{4}} - 1}{2^{\frac{7}{6}} - 1}.$$

The associated operator acting by convolution against the Fourier transform of μ_ζ is denoted by T_ζ. For $\zeta = 1/3$ we see that

$$\mu_\zeta = \sum_{1 \ll \lambda \leqslant \delta_0^{-6}} \nu_{\delta,Ai}^{\lambda},$$

which means, by interpolation, that it is sufficient to prove

$$\begin{aligned}\|T_{it}\|_{L_{x_3}^{2/(2-\tilde{\sigma})}(L^1_{(x_1,x_2)}) \to L_{x_3}^{2/\tilde{\sigma}}(L^\infty_{(x_1,x_2)})} &\lesssim 1,\\ \|T_{1+it}\|_{L^2 \to L^2} &\lesssim 1,\end{aligned}$$

with constants uniform in $t \in \mathbb{R}$.

In order to prove the first estimate, we need the decay bound (2.3.2), i.e.,

$$|\widehat{\mu_{it}}(\xi)| \lesssim \frac{1}{(1+|\xi_3|)^{1/4}}.$$

This follows right away by using the estimate on the Fourier transform of $\nu_{\delta,Ai}^\lambda$, the definition of μ_ζ, and the fact that each $\widehat{\nu_{\delta,Ai}^\lambda}$ has its support located at $(\lambda, \lambda, \lambda)$.

We prove the second $L^2 \to L^2$ estimate by using Lemma 2.2.5. We need to prove

$$\Bigg\| \sum_{1 \ll \lambda \leqslant \delta_0^{-6}} \lambda^{-\frac{7}{6} - \frac{7}{4}it} \, \nu_{\delta,Ai}^\lambda \Bigg\|_{L^\infty} \lesssim \frac{1}{\left| 2^{-\frac{7}{4}it} - 1 \right|}, \tag{3.4.6}$$

uniformly in t.

As in [51, Subsection 5.1.1] we now apply Fourier inversion using the formulas (3.4.4), (3.4.5), and the form of the integral from Lemma 2.2.2, (a). Then after changing coordinates in the integration from (ξ_1, ξ_2, ξ_3) to (s_1, s_2, s_3) one gets

$$\begin{aligned} \nu_{\delta,Ai}^\lambda(x) = &\lambda^{13/6} \int e^{-i\lambda s_3 (B_0(s',\delta,\sigma) - s_1 x_1 - s_2 x_2 - x_3)} \chi_0(\lambda^{2/3} B_1(s', \delta, \sigma)) \\ & \times g(\lambda^{2/3} B_1(s', \delta, \sigma), \lambda^{-1/3}, \delta, \sigma, s) \tilde{\chi}_1(s) \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3, \end{aligned}$$

where g is the smooth function from Lemma 2.2.2, (a), whose derivatives of any order are uniformly bounded, and where

$$\tilde{\chi}_1(s) := \chi_1(s_1 s_3) \chi_1(s_2 s_3) \chi_1(s_3) s_3^2.$$

We may now also restrict ourselves to the situation where $|x| \lesssim 1$, since otherwise we can get a factor λ^{-N} by integrating by parts.

Finally, we change coordinates from $s' = (s_1, s_2)$ to (z, s_2), where $z := \lambda^{2/3} B_1(s', \delta, \sigma)$, and so by Lemma 3.4.1 we have

$$z = \lambda^{2/3}(-s_1 + s_2^{(n-1)/(n-2)} G_3(s_2, \delta, \sigma)),$$

that is

$$s_1 = s_2^{(n-1)/(n-2)} G_3(s_2, \delta, \sigma) - \lambda^{-2/3} z.$$

Thus we obtain

$$\begin{aligned} \nu_{\delta,Ai}^\lambda(x) = &\lambda^{3/2} \int e^{-i\lambda s_3 \Phi(z, s_2, x, \delta, \sigma)} \\ & \times g\Big(z, \lambda^{-1/3}, \delta, \sigma, s_2^{(n-1)/(n-2)} G_3(s_2, \delta, \sigma) - \lambda^{-2/3} z, s_2, s_3\Big) \\ & \times \tilde{\chi}_1\Big(s_2^{(n-1)/(n-2)} G_3(s_2, \delta, \sigma) - \lambda^{-2/3} z, s_2\Big) \chi_0(z) \mathrm{d}z \mathrm{d}s_2 \mathrm{d}s_3, \end{aligned} \tag{3.4.7}$$

where by using the expressions for $B_0(s', \delta, \sigma)$ and $G_5(s_2, \delta, \sigma)$ from Lemma 3.4.1 one gets

$$\begin{aligned} \Phi(z, s_2, x, \delta, \sigma) := &s_2^{n/(n-2)} G_5(s_2, \delta, \sigma) - s_2^{(n-1)/(n-2)} G_3(s_2, \delta, \sigma) x_1 - s_2 x_2 - x_3 \\ & + \lambda^{-2/3} z(x_1 - s_2^{1/(n-2)} G_1(s_2, \delta, \sigma)). \end{aligned} \tag{3.4.8}$$

We may shorten the expression in (3.4.7) to

$$\begin{aligned}\nu^{\lambda}_{\delta,Ai}(x) = \lambda^{3/2}\int & e^{-i\lambda s_3\Phi(z,s_2,x,\delta,\sigma)} \\ & \times \tilde{g}(z, s_2^{1/(n-2)}, s_3, \lambda^{-1/3}, \delta, \sigma)\mathrm{d}z\mathrm{d}s_2\mathrm{d}s_3,\end{aligned} \tag{3.4.9}$$

where $\tilde{g}$ is smooth with uniformly bounded derivatives and localising the integration domain to $|z| \lesssim 1$, $s_2 \sim |s_3| \sim 1$.

Next, we notice that $\Phi(z, s_2^{n-2}, x, 0, \sigma)$ is a polynomial in s_2 by (3.4.3). We therefore substitute $s_0 = s_2^{1/(n-2)}$ and denote

$$\tilde{\Phi}(z, s_0, x, \delta, \sigma) = \Phi(z, s_0^{1/(n-2)}, x, \delta, \sigma).$$

We are interested in localising the integration in (3.4.9) to the place where $\partial^2_{s_0}\tilde{\Phi} = 0$ and $\partial^3_{s_0}\tilde{\Phi} \neq 0$. In order to carry out this reduction we need another simple lemma. It will be applied to the first three terms of

$$\begin{aligned}\tilde{\Phi}(z, s_0, x, 0, \sigma) =& s_0^n G_5(s_0^{n-2}, 0, \sigma) - s_0^{n-1}G_3(s_0^{n-2}, 0, \sigma)x_1 - s_0^{n-2}x_2 - x_3 \\ & + \lambda^{-2/3}z(x_1 - s_0 G_1(s_0^{n-2}, 0, \sigma)),\end{aligned}$$

which constitute a polynomial in s_0 whose derivatives have at most two zeros not located at the origin. Note that the last term in the above expression is arbitrarily small.

Lemma 3.4.2. *Assume $n \geqslant 5$ and consider a number $x_0 \sim 1$. Let us define a polynomial of the form*

$$P(x) := x^{n-2}(x^2 + bx + c) = x^n + bx^{n-1} + cx^{n-2}$$

whose second derivative can be written as

$$P''(x) := n(n-1)x^{n-4}(x - x_0 + \varepsilon)(x - x_0 - \varepsilon).$$

If $|\varepsilon| \leqslant c_1$ for a sufficiently small constant c_1, then $|P'(x)| \sim 1$ on a neighbourhood of x_0, which depends on c_1, but not on ε. On the other hand, if $|\varepsilon| > c_2$ for some $c_2 > 0$ and $x_0 - \varepsilon \sim 1$ (resp. $x_0 + \varepsilon \sim 1$), then $|P'''(x_0 - \varepsilon)| \sim_{c_2} 1$ (resp. $|P'''(x_0 + \varepsilon)| \sim_{c_2} 1$).

Proof. One needs to express b and c in terms of x_0 and ε, after which it is easy to prove the lemma by a straightforward calculation. □

From the first conclusion of Lemma 3.4.2 we see that if the zeros of $\partial^2_{s_0}\tilde{\Phi}$ which are away from the origin are too close to each other, then we may use stationary phase or integration by parts to obtain a factor of $\lambda^{-1/2}$ (or better) and so the left hand side of (3.4.6) is absolutely summable. Therefore we may assume that there is at least some distance between the zeros of $\partial^2_{s_0}\tilde{\Phi}$. From the second conclusion of Lemma 3.4.2 we obtain $|\partial^3_{s_0}\tilde{\Phi}| \sim 1$ in a neighbourhood of those zeros within the integration domain (i.e., for those located at ~ 1).

Therefore, we may now use the implicit function theorem and obtain a parametrisation of a zero of the first three terms of $\partial_{s_0}^2 \tilde{\Phi}$:

$$\partial_{s_0}^2 (s_0^n G_5(s_0^{n-2}, \delta, \sigma) - s_0^{n-1} G_3(s_0^{n-2}, \delta, \sigma) x_1 - s_0^{n-2} x_2),$$

which we shall denote by $s_0^c(x, \delta, \sigma)$, and assume it is located away from the origin. All such zeros can be treated the same way.

We may assume we integrate arbitrarily near the zero $s_0^c(x, \delta, \sigma)$ since again we could otherwise use stationary phase or integration by parts. We may then use a Taylor approximation for the first three terms in $\tilde{\Phi}$ at $s_0^c(x, \delta, \sigma)$ and obtain after translating $s_0 \mapsto s_0 + s_0^c$ that the phase has the form

$$\begin{aligned} \tilde{\Phi}_1(z, s_0, x, \delta, \sigma) =& \tilde{B}_0(x, \delta, \sigma) - \tilde{B}_1(x, \delta, \sigma) s_0 + \tilde{B}_3(s_0, x, \delta, \sigma) s_0^3 \\ &+ \lambda^{-2/3} z \tilde{G}_1(s_0, x, \delta, \sigma) - \lambda^{-2/3} z \tilde{G}_2(s_0, x, \delta, \sigma) s_0 \end{aligned}$$

with functions $\tilde{B}_i$, $i = 0, 1, 3$, being smooth and $|\tilde{B}_3| \sim 1$. The functions $\tilde{G}_i$ are also smooth and have the property that they do not depend on s_0 when $\delta = 0$. Note also $\tilde{G}_2(s_0, x, 0, \sigma) = 1$.

Hence, we have obtained an Airy type integral with an error term of size at most $\lambda^{-2/3}$. We denote this newly obtained function by $\tilde{\nu}_{\delta,Ai}^{\lambda}$:

$$\begin{aligned} \tilde{\nu}_{\delta,Ai}^{\lambda}(x) = \lambda^{3/2} \int & e^{-i\lambda s_3 \tilde{\Phi}_1(z, s_0, x, \delta, \sigma)} \\ & \times \tilde{g}_1(z, s_0, s_3, \lambda^{-1/3}, \delta, \sigma) \mathrm{d}z \mathrm{d}s_0 \mathrm{d}s_3, \end{aligned}$$

where $\tilde{g}_1$ has the same properties as $\tilde{g}$, except that now the integration is over the domain where $|z| \lesssim 1$, $|s_3| \sim 1$, and $|s_0| \ll 1$.

We now prove (3.4.6) for the remaining piece $\tilde{\nu}_{\delta,Ai}^{\lambda}$. Let us begin with the case when

$$A := \lambda^{2/3} \tilde{B}_1(x, \delta, \sigma)$$

satisfies $|A| \gg 1$. We claim that in this case we can estimate the function $\tilde{\nu}_{\delta,Ai}^{\lambda}$ by $\lambda^{7/6} |A|^{-1/4}$, which is absolutely summable in λ in the expression (3.4.6) for μ_{1+it}. We need a modification of Lemma 2.2.2, (b).

Lemma 3.4.3. *Consider the integral*

$$\int e^{i\lambda(-b_1 s_0 + b_3(s_0) s_0^3 + \lambda^{-2/3} g(s_0))} a_0(s_0, \lambda^{-1/3}) \chi_0(s_0) \mathrm{d}s_0,$$

where all the appearing functions are smooth with uniformly bounded derivatives, and $|b_3(s_0)| \sim 1$. *This integral can be estimated up to a constant by* $\lambda^{-1/3} |\lambda^{2/3} b_1|^{-1/4}$ *if* $|\lambda^{2/3} b_1| \gg 1$, $\lambda \gg 1$, *and* χ_0 *is supported in a sufficiently small neighbourhood of the origin.*

Proof. Without loss of generality we may assume $b_3 > 0$. We proceed similarly as in the proof of Lemma 2.2.2, (b). The main point is that since we may assume $|b_1| \gg |\lambda^{-2/3} g^{(k)}|$

for finitely many $k \geqslant 0$, the term $\lambda^{-2/3} g(s_0)$ will not have any significant influence. The first derivative of the phase is

$$i\lambda(-b_1 + 3b_3(s_0)s_0^2 + b_3'(s_0)s_0^3 + \lambda^{-2/3} g'(s_0)),$$

and hence if $b_1 < 0$ or $|b_1| \gtrsim 1$, then the phase has no critical points since the first two terms are dominant, and its derivative is of size $\gtrsim |\lambda b_1|$. Using integration by parts we get the estimate $|\lambda b_1|^{-1}$.

Therefore we may assume $0 < b_1 \ll 1$ and substitute $b_1^{1/2} s_0$ to obtain

$$\begin{aligned} b_1^{1/2} \int & e^{i\lambda b_1^{3/2}(-s_0 + b_3(b_1^{1/2}s_0)s_0^3 + b_1^{-3/2}\lambda^{-2/3} g(b_1^{1/2}s_0))} \\ & \times a_0(b_1^{1/2}s_0, \lambda^{-1/3})\chi_0(b_1^{1/2}s_0)\mathrm{d}s_0. \end{aligned}$$

One can now easily check that the function

$$s_0 \mapsto -s_0 + b_3(b_1^{1/2}s_0)s_0^3 + b_1^{-3/2}\lambda^{-2/3} g(b_1^{1/2}s_0)$$

has precisely two critical points near ± 1. Near these critical points the second derivative is of size ~ 1 and so by stationary phase one gets the bound $b_1^{1/2}|\lambda b_1^{3/2}|^{-1/2} = \lambda^{-1/3}|\lambda^{2/3} b_1|^{-1/4}$. Away from the critical points the size of the derivative of the phase is $\sim \lambda b_1^{3/2} \max\{s_0^2, 1\}$, and so integrating by parts one gets the estimate $|\lambda b_1|^{-1}$. $\square$

Therefore after one applies the above lemma, our problem is reduced to the case $|A| \lesssim 1$. Our next step is to substitute $s_0 \mapsto \lambda^{-1/3} s_0$. Then one gets

$$\begin{aligned} \tilde{\nu}_{\delta,Ai}^{\lambda}(x) = \lambda^{7/6} \int & e^{-i\lambda s_3 \tilde{\Phi}_1(z,\lambda^{-1/3}s_0,x,\delta,\sigma)} \\ & \times \tilde{g}_1(z, \lambda^{-1/3}s_0, s_3, \lambda^{-1/3}, \delta, \sigma)\mathrm{d}z\mathrm{d}s_0\mathrm{d}s_3, \end{aligned}$$

where

$$\begin{aligned} \lambda\tilde{\Phi}_1(z, \lambda^{-1/3}s_0, x, \delta, \sigma) =& \lambda\tilde{B}_0(x,\delta,\sigma) - As_0 + \tilde{B}_3(\lambda^{-1/3}s_0, x, \delta, \sigma)s_0^3 \\ & + \lambda^{1/3} z\tilde{G}_1(\lambda^{-1/3}s_0, x, \delta, \sigma) - z\tilde{G}_2(\lambda^{-1/3}s_0, x, \delta, \sigma)s_0, \end{aligned}$$

and the new integration domain is $|z| \lesssim 1$, $|s_3| \sim 1$, and $|s_0| \ll \lambda^{1/3}$.

Using a Taylor approximation we can rewrite the $\tilde{G}_1$ term as

$$\tilde{G}_1(\lambda^{-1/3}s_0, x, \delta, \sigma) = \tilde{G}_1(0, x, \delta, \sigma) + \lambda^{-1/3}s_0 r(\lambda^{-1/3}s_0, x, \delta, \sigma).$$

where $|\partial_t^N r(t, x, \delta, \sigma)| \ll_N 1$ for any $N \geqslant 0$ since $\tilde{G}_1$ is constant when $\delta = 0$. Therefore, if we denote $\tilde{G}_3 = \tilde{G}_2 - r$, then $\tilde{G}_3$ has the same properties as $\tilde{G}_2$ (in particular $\tilde{G}_3 \sim 1$), and we can write

$$\begin{aligned} \lambda\tilde{\Phi}_1(z, \lambda^{-1/3}s_0, x, \delta, \sigma) =& \lambda\tilde{B}_0(x,\delta,\sigma) - As_0 + \tilde{B}_3(\lambda^{-1/3}s_0, x, \delta, \sigma)s_0^3 \\ & + \lambda^{1/3} z\tilde{G}_1(0, x, \delta, \sigma) - z\tilde{G}_3(\lambda^{-1/3}s_0, x, \delta, \sigma)s_0. \end{aligned}$$

From this expression one sees that we can get an integrable factor of size $(1+|s_0|^2)^{-N/2}$ in the amplitude of $\tilde{\nu}^\lambda_{\delta,Ai}$ by using integration by parts in s_0, i.e., we can assume

$$\left|\partial_z^{\alpha_1}\partial_{s_0}^{\alpha_2}\partial_{s_3}^{\alpha_3}\Big(\tilde{g}_1(z,\lambda^{-1/3}s_0,s_3,\lambda^{-1/3},\delta,\sigma)\Big)\right| \lesssim C_{\alpha_1,\alpha_2,\alpha_3}(1+|s_0|^2)^{-N/2},$$

as the unbounded terms in the expression for the s_0 derivative of $\lambda\tilde{\Phi}_1(z,\lambda^{-1/3}s_0,x,\delta,\sigma)$ vanish.

Let us denote by

$$E := \lambda\tilde{B}_0(x,\delta,\sigma), \qquad\qquad F := \lambda^{1/3}\tilde{G}_1(0,x,\delta,\sigma),$$

the unbounded terms of the phase. We need to reduce our problem to the case when $|E| \lesssim 1$ and $|F| \lesssim 1$ since then we can simply apply the oscillatory sum lemma.

We begin with the case $|F| \gg 1$. Let us consider the z integration. The factor tied with z in the phase is

$$F - \tilde{G}_3(\lambda^{-1/3}s_0,x,\delta,\sigma)s_0 = F - \tilde{G}_3 s_0,$$

where $\tilde{G}_3(\lambda^{-1/3}s_0,x,\delta,\sigma) \sim 1$. We may therefore assume we are integrating over the area in s_0 where

$$|F - \tilde{G}_3 s_0| \lesssim |F|^\varepsilon,$$

since otherwise we can use integration by parts in z and gain a factor $|F|^{-\varepsilon}$. In particular, in this case we have $|s_0| \sim |F|$. But then the integrable factor $(1+|s_0|^2)^{-N/2}$ is of size $|F|^{-N}$ and so we obtain the required bound.

It remains to consider the case $|F| \lesssim 1$ and $|E| \gg 1$. The idea in this case is to use integration by parts in s_3, which enables us to localize the integration to the set where $|\lambda\tilde{\Phi}_1| \lesssim |E|^\varepsilon$. If we now take $|E|$ sufficiently large compared to both $|A|$ and $|F|$, then we see that $|\lambda\tilde{\Phi}_1| \lesssim |E|^\varepsilon$ forces $|s_0| \sim |E|^{1/3}$. But this implies that the integrable factor $(1+|s_0|^2)^{-N/2}$ is of size $|E|^{-N/3}$, which is what we wanted. We are done with the part near the Airy cone.

3.4.3 Estimates away from the Airy cone – first considerations

Recall from (3.4.4) and (3.4.5) that we may write

$$\begin{aligned}\widehat{\nu_{\delta,l}^\lambda}(\xi) =& \lambda^{-1/2}\chi_1((2^{-l}\lambda)^{2/3}B_1(s',\delta,\sigma))\\ &\chi_1(s_1s_3)\chi_1(s_2s_3)\chi_1(s_3)e^{-i\lambda s_3 B_0(s',\delta,\sigma)}\\ &\int e^{-i\lambda s_3(B_3(s_2,\delta,\sigma,y_1)y_1^3 - B_1(s',\delta,\sigma)y_1)}a_0(y_1,s,\delta)\chi_0(y_1)\mathrm{d}y_1,\end{aligned}$$

where $1 \ll 2^l \ll \lambda$. Applying Lemma 2.2.2, (b), we obtain

$$\begin{aligned}\widehat{\nu_{\delta,l}^{\lambda}}(\xi) =& \lambda^{-1/2}\chi_1((2^{-l}\lambda)^{2/3}B_1(s',\delta,\sigma))\\ &\chi_1(s_1s_3)\chi_1(s_2s_3)\chi_1(s_3)e^{-i\lambda s_3 B_0(s',\delta,\sigma)}\\ &\Big(s_3^{-1/2}\lambda^{-1/2}|B_1(s',\delta,\sigma)|^{-1/4}\\ &\times a(|B_1(s',\delta,\sigma)|^{1/2}, s; s_3\lambda|B_1(s',\delta,\sigma)|^{3/2})\, e^{is_3\lambda|B_1(s',\delta,\sigma)|^{3/2}q(|B_1(s',\delta,\sigma)|^{1/2},s_2)}\\ &+ (s_3\lambda|B_1(s',\delta,\sigma)|)^{-1}\, E(s_3\lambda|B_1(s',\delta,\sigma)|^{3/2}, |B_1(s',\delta,\sigma)|^{1/2}, s)\Big),\end{aligned}$$

where we have slightly simplified the situation by ignoring the sign of the function q since both q_+ and q_- appearing in Lemma 2.2.2, (b), can be treated in the same way. Note that q depends in the second variable only in s_2 and not s since the same is true for B_3, as can be readily seen from the proof of Lemma 2.2.2, (b). Recall that a, q, and E are smooth, and $|q| \sim 1$. E and all its derivatives have Schwartz decay in the first variable, and a is a classical symbol of order 0 in the $s_3\lambda|B_1(s',\delta,\sigma)|^{3/2}$ variable.

We denote

$$z = (2^{-l}\lambda)^{2/3}B_1(s',\delta,\sigma),$$

and slightly change a and E in order to absorb the s_3 factors. Then we can rewrite the previous expression for $\widehat{\nu_{\delta,l}^{\lambda}}$ as

$$\begin{aligned}\widehat{\nu_{\delta,l}^{\lambda}}(\xi) =& \lambda^{-1/2}\chi_1(z)\chi_1(s_1s_3)\chi_1(s_2s_3)\chi_1(s_3)e^{-i\lambda s_3 B_0(s',\delta,\sigma)}\\ &\Big(\lambda^{-1/2}(2^{-l}\lambda)^{1/6}|z|^{-1/4}\\ &\times a((2^l\lambda^{-1})^{1/3}|z|^{1/2}, s; 2^l|z|^{3/2})\, e^{-is_3 2^l|z|^{3/2}q((2^l\lambda^{-1})^{1/3}|z|^{1/2},s_2)}\\ &+ \lambda^{-1}(2^{-l}\lambda)^{2/3}|z|^{-1}\, E(2^l|z|^{3/2}, (2^l\lambda^{-1})^{1/3}|z|^{1/2}, s)\Big).\end{aligned}$$

From this we easily see that

$$\|\widehat{\nu_{\delta,l}^{\lambda}}\|_{L^\infty} \lesssim \lambda^{-5/6}2^{-l/6}.$$

We plan to use complex interpolation and the two parameter oscillatory sum lemma (Lemma 2.2.7). We consider the following function parametrized by $\zeta \in \mathbb{C}$:

$$\mu_\zeta = \gamma(\zeta) \sum_{\substack{1 \ll \lambda \leqslant \delta_0^{-6}\\ M_0 \leqslant 2^l \leqslant \lambda/M_1}} \lambda^{\frac{7-21\zeta}{12}}\, 2^{\frac{1-3\zeta}{6}l}\, \nu_{\delta,l}^{\lambda},$$

for an appropriate $\gamma(\zeta)$ to be chosen later as in (2.2.3). We shall also use the one parameter oscillatory sum lemma for certain subcases, and therefore we shall need to add appropriate factors to γ of the form 2.2.1. The operator associated to μ_ζ we denote by T_ζ.

For $\zeta = 1/3$ we see that

$$\mu_\zeta = \sum_{\substack{1 \ll \lambda \leqslant \delta_0^{-6}\\ M_0 \leqslant 2^l \leqslant \lambda/M_1}} \nu_{\delta,l}^{\lambda},$$

which means, by Stein's interpolation theorem, that it is sufficient to prove

$$\|T_{it}\|_{L^{2/(2-\tilde{\sigma})}_{x_3}(L^1_{(x_1,x_2)}) \to L^{2/\tilde{\sigma}}_{x_3}(L^\infty_{(x_1,x_2)})} \lesssim 1,$$
$$\|T_{1+it}\|_{L^2 \to L^2} \lesssim 1,$$

with constants uniform in $t \in \mathbb{R}$.

In order to prove the first estimate we need the decay bound (2.3.2), i.e.,

$$|\widehat{\mu_{it}}(\xi)| \lesssim \frac{1}{(1+|\xi_3|)^{1/4}}.$$

This bound follows easily by the L^∞ bound on the Fourier transform of $\nu^\lambda_{\delta,l}$, the definition of μ_ζ, and the fact that each $\widehat{\nu^\lambda_{\delta,l}}$ has its support located at $(\lambda, \lambda, \lambda)$.

It remains to prove the $L^2 \to L^2$ estimate

$$\Bigg\| \sum_{\substack{1 \ll \lambda \leqslant \delta_0^{-6} \\ M_0 \leqslant 2^l \leqslant \lambda/M_1}} \lambda^{-\frac{7}{6}-\frac{7}{4}it} 2^{-\frac{1}{3}l-\frac{1}{2}ilt} \nu^\lambda_{\delta,l} \Bigg\|_{L^\infty} \lesssim \frac{1}{\big|\gamma(1+it)\big|}, \tag{3.4.10}$$

uniformly in t.

We split the function $\nu^\lambda_{\delta,l}$ as

$$\nu^\lambda_{\delta,l} = \nu^E_{\lambda,l} + \nu^a_{\lambda,l},$$

where

$$\begin{aligned}\widehat{\nu^E_{\lambda,l}}(\xi) =& \lambda^{-5/6} 2^{-Nl} \tilde{\chi}_1(s,z) e^{-i\lambda s_3 B_0(s',\delta,\sigma)} \\ &\times E(2^l|z|^{3/2}, (2^l\lambda^{-1})^{1/3}|z|^{1/2}, s)\end{aligned}$$

and

$$\begin{aligned}\widehat{\nu^a_{\lambda,l}}(\xi) =& \lambda^{-5/6} 2^{-l/6} \tilde{\chi}_1(s,z) e^{-i\lambda s_3 B_0(s',\delta,\sigma)} \\ &\times a((2^l\lambda^{-1})^{1/3}|z|^{1/2}, s; 2^l|z|^{3/2})\, e^{-is_3 2^l|z|^{3/2} q((2^l\lambda^{-1})^{1/3}|z|^{1/2}, s_2)},\end{aligned}$$

with appropriate (and in each of the above expressions possibly different) $\tilde{\chi}_1$ smooth cutoff functions localising to the area where $|s_1| \sim s_2 \sim |s_3| \sim |z| \sim 1$. In the expression for $\nu^E_{\lambda,l}$ we obtain the factor 2^{-Nl} by using the Schwartz property in the first variable of E, and so the function E is slightly different than before, but with the same properties.

3.4.4 Estimates away from the Airy cone – the estimate for $\nu^E_{\lambda,l}$

The function $\nu^E_{\lambda,l}$ can be treated similarly as the function $\nu^\lambda_{\delta,Ai}$ in the case near the Airy cone. We first apply the inverse of the Fourier transform to $\widehat{\nu^E_{\lambda,l}}$, and then substitute $s = (s_1, s_2, s_3)$ for $\xi = (\xi_1, \xi_2, \xi_3)$. Recall that $z = (2^{-l}\lambda)^{2/3} B_1(s', \delta, \sigma)$ and so by Lemma 3.4.1 one has

$$s_1 = s_2^{(n-1)/(n-2)} G_3(s_2, \delta, \sigma) - (2^l\lambda^{-1})^{2/3} z.$$

We plug in this expression for s_1 and also substitute s_0 for $s_2^{1/(n-2)}$. In the end one gets

$$\begin{aligned}\nu^E_{\lambda,l}(x) =&\lambda^{3/2}2^{-Nl}\int e^{-i\lambda s_3\Phi_2(z,s_0,x,\delta,\sigma)}\\ &\times g_2\Big(2^l,(2^l\lambda^{-1})^{1/3},z,s_0,s_3,\delta,\sigma\Big)\mathrm{d}z\mathrm{d}s_0\mathrm{d}s_3,\end{aligned}$$

where g_2 is smooth and has all of its derivatives Schwartz in the first variable, and where

$$\begin{aligned}\Phi_2(z,s_0,x,\delta,\sigma) :=&s_0^nG_5(s_0^{n-2},\delta,\sigma)-s_0^{n-1}G_3(s_0^{n-2},\delta,\sigma)x_1-s_0^{n-2}x_2-x_3\\ &+(2^l\lambda^{-1})^{2/3}z(x_1-s_0G_1(s_0^{n-2},\delta,\sigma)).\end{aligned}$$

The only difference compared to the phase in (3.4.8) is that there $|z|\lesssim 1$, while here $|z|\sim 1$, and instead of the $\lambda^{-2/3}$ factor in front of z in the phase in (3.4.8), here we have the much larger factor $(2^l\lambda^{-1})^{2/3}$.

We can now reduce to the situation where $|x|\lesssim 1$. Namely, if $|x_1|\gg 1$ then we integrate by parts in z to gain a factor $(\lambda(2^l\lambda^{-1})^{2/3})^{-N}$. Otherwise if $|x_1|\lesssim 1$ and $|x_2|\gg 1$, then we integrate by parts in s_0 to obtain a factor λ^{-N}, and if $|(x_1,x_2)|\lesssim 1$ and $|x_3|\gg 1$, we integrate by parts in s_3 to again gain a factor of λ^{-N}.

Next, recall that $(2^l\lambda^{-1})^{2/3}\ll 1$. Therefore, we may use again Lemma 3.4.2 and argue similarly as we did in the case near the Airy cone to reduce ourselves to a small neighbourhood of a point where the second derivative in s_0 of the first three terms of Φ_2 vanishes and $|\partial^3_{s_0}\Phi_2|\sim 1$. By the implicit function theorem we may parametrize this point as $s^c=s^c(x,\delta,\sigma)$:

$$\partial^2_{s_0}\big|_{s_0=s^c}\,(s_0^nG_5(s_0^{n-2},\delta,\sigma)-s_0^{n-1}G_3(s_0^{n-2},\delta,\sigma)x_1-s_0^{n-2}x_2-x_3)=0.$$

The point s^c depends smoothly on (x,δ,σ).

Translating to the point s^c and localising to a small neighbourhood we obtain a new function $\tilde\nu^E_{\lambda,l}$ of the form

$$\begin{aligned}\tilde\nu^E_{\lambda,l}(x) =&\lambda^{3/2}2^{-Nl}\int e^{-i\lambda s_3\Phi_2(z,s_0,x,\delta,\sigma)}\\ &\times \tilde g_2\Big(2^l,(2^l\lambda^{-1})^{1/3},z,s_0,s_3,\delta,\sigma\Big)\mathrm{d}z\mathrm{d}s_0\mathrm{d}s_3,\end{aligned}$$

where $\tilde g_2$ has the same properties as g_2, except that now $|s_0|\ll 1$. The new phase is

$$\begin{aligned}\tilde\Phi_2(z,s_0,x,\delta,\sigma) =&\tilde B_0(x,\delta,\sigma)-\tilde B_1(x,\delta,\sigma)s_0+\tilde B_3(s_0,x,\delta,\sigma)s_0^3\\ &+(2^l\lambda^{-1})^{2/3}zH_0(s_0,x,\delta,\sigma)-(2^l\lambda^{-1})^{2/3}zH_1(s_0,x,\delta,\sigma)s_0,\end{aligned}$$

where $|\tilde B_3|\sim 1$ and $H_1\sim 1$. Additionally, one can see that H_0 and H_1 do not depend on s_0 when $\delta=0$.

The next step is to develop the whole phase $\tilde\Phi_2$ at the point where $\partial^2_{s_0}\tilde\Phi_2=0$. The reason for this is that the factor $(2^l\lambda^{-1})^{2/3}$ is too large, and we cannot apply something similar to Lemma 3.4.3. Let us denote the critical point of $\partial_{s_0}\tilde\Phi_2$ by $s_0^c=s_0^c(x,\delta,\sigma,(2^l\lambda^{-1})^{2/3}z)$.

Note that s_0^c is identically 0 when either $\delta = 0$ or the variable refering to $(2^l\lambda^{-1})^{2/3}z$ is 0. Therefore, we can actually write

$$s_0^c = (2^l\lambda^{-1})^{2/3} z\, \tilde{s}_0^c(x,\delta,\sigma,(2^l\lambda^{-1})^{2/3}z),$$

where $\tilde{s}_0^c$ is smooth and identically 0 when $\delta = 0$.

If we shorten $\rho = (2^l\lambda^{-1})^{2/3}z$, then the expression for the first derivative of $\tilde{\Phi}_2$ at the point s_0^c has the form

$$\begin{aligned}\partial_{s_0}\tilde{\Phi}_2(z,s_0^c,x,\delta,\sigma) &= (s_0^c)^2\, b(s_0^c,x,\delta,\sigma) - \rho h(s_0^c,x,\delta,\sigma) - \tilde{B}_1(x,\delta,\sigma)\\ &= \rho^2(\tilde{s}_0^c)^2\, b(s_0^c,x,\delta,\sigma) - \rho h(s_0^c,x,\delta,\sigma) - \tilde{B}_1(x,\delta,\sigma),\end{aligned}$$

where $h(s_0^c,x,\delta,\sigma) \sim 1$ and $|b(s_0^c,x,\delta,\sigma)| \sim 1$ for some smooth functions h and b.

One can easily check that $|\partial_{s_0}^3\tilde{\Phi}_2(z,s_0,x,\delta,\sigma)| \sim 1$. Therefore, developing the phase $\tilde{\Phi}_2$ at the point s_0^c, we may write

$$\tilde{\Phi}_3(z,s_0,x,\delta,\sigma) = b_0(\rho) - \Big[b_1 + \rho\tilde{b}_1(\rho)\Big]s_0 + b_3(s_0,\rho)s_0^3, \tag{3.4.11}$$

where we suppressed the dependence of $b_0, b_1, \tilde{b}_1$, and b_3 on the bounded parameters (x,δ,σ). Here we know that $\tilde{b}_1 \sim 1$ and $|b_3| \sim 1$. We may again assume $|s_0| \ll 1$ as on the other part where $|s_0| \gtrsim 1$ we could use integration by parts or stationary phase and obtain an expression which when plugged into (3.4.10) would be absolutely summable in both λ and 2^l.

Finally, we develop the term b_0 at 0 and substitute $s_0 \mapsto \lambda^{-1/3}s_0$. Then

$$\begin{aligned}\lambda\tilde{\Phi}_3(z,s_0,x,\delta,\sigma) &= \lambda\Big(b_0^0 + \rho b_0^1 + \rho^2\tilde{b}_0(\rho) - \lambda^{-1/3}\Big[b_1 + \rho\tilde{b}_1(\rho)\Big]s_0 + \lambda^{-1}b_3(\lambda^{-1/3}s_0,\rho)s_0^3\Big)\\ &= \lambda b_0^0 + \lambda^{1/3}2^{2l/3}b_0^1 z + \lambda^{-1/3}2^{4l/3}\tilde{b}_0(\rho)z^2\\ &\quad - \Big[\lambda^{2/3}b_1 + 2^{2l/3}\tilde{b}_1(\rho)\, z\Big]s_0 + b_3(\lambda^{-1/3}s_0,\rho)s_0^3,\end{aligned}$$

and the remaining part of the function $\tilde{\nu}_{\lambda,l}^E$ is of the form

$$\begin{aligned}\tilde{\tilde{\nu}}_{\lambda,l}^E(x) =& \lambda^{7/6}2^{-Nl}\int e^{-i\lambda s_3\tilde{\Phi}_3(z,\lambda^{-1/3}s_0,x,\delta,\sigma)}\\ &\times g_3\Big(2^l,(2^l\lambda^{-1})^{1/3},z,\lambda^{-1/3}s_0,s_3,\delta,\sigma\Big)\mathrm{d}z\mathrm{d}s_0\mathrm{d}s_3,\end{aligned} \tag{3.4.12}$$

where again g_3 has the same properties as $\tilde{g}_2$ and in the area of integration we have $|s_0| \ll \lambda^{1/3}$.

Now, we first note that we can assume $\lambda^{-1/3}2^{4l/3} \ll 1$ since otherwise we can easily sum in both λ and l using the factor 2^{-Nl} for a sufficiently large N. Next, we introduce

$$A := \lambda b_0^0, \qquad B := \lambda^{1/3}2^{2l/3}b_0^1, \qquad D := \lambda^{2/3}b_1.$$

We need to reduce our problem to the situation when A, B, and D are bounded since then we can simply apply the (one parameter) oscillatory sum lemma. When this is the case, the size of the integration domain in (3.4.12) is not a problem since, if we split the integration domain to the areas where $|s_0| \lesssim 2^{l/3}$ and $|s_0| \gg 2^{l/3}$, the first part has domain size $2^{l/3}$, which is admissible, and in the second part the amplitude is integrable in s_0 after using integration by parts.

Case $|D| \gg 1$. We consider two subcases. The first subcase is when

$$|\lambda^{2/3} b_1 + 2^{2l/3} \tilde{b}_1(\rho)\, z| = |D + 2^{2l/3} \tilde{b}_1(\rho)\, z| > 1.$$

Here we can actually use the Airy integral lemma (Lemma 2.2.2, (b)) applied to s_0 integration before substituting $s_0 \mapsto \lambda^{-1/3} s_0$, i.e., using the phase form (3.4.11), and obtain the bound

$$\|\tilde{\nu}^E_{\lambda,l}\|_{L^\infty} \lesssim \lambda^{7/6} 2^{-lN} \int \chi_1(z) |D + 2^{2l/3} \tilde{b}_1(\rho)\, z|^{-\varepsilon} \mathrm{d}z,$$

for some constant $\varepsilon > 0$. After plugging into (3.4.10) this is absolutely summable in λ. Namely, in the cases $|D| \ll 2^{2l/3}$ and $|D| \gg 2^{2l/3}$ we get the estimate $|D|^{-\varepsilon}$, which is summable, and the case $|D| \sim 2^{2l/3}$ happens for only $\mathcal{O}(1)$ λ's, which depend on l.

The second subcase is when

$$|D + 2^{2l/3} \tilde{b}_1(\rho)\, z| \leqslant 1.$$

Then necessarily again $|D| \sim |2^{2l/3}|$, and this can happen only for $\mathcal{O}(1)$ λ's. By (3.4.12) we have

$$\|\tilde{\nu}^E_{\lambda,l}\|_{L^\infty} \lesssim \lambda^{7/6} 2^{-lN},$$

for maybe some different N. The factor $\lambda^{7/6}$ is retained since in this case we can get an integrable factor in s_0 by using integration by parts. After plugging into (3.4.10) we may sum over the $\mathcal{O}(1)$ λ's and then in l.

Case $|D| \lesssim 1$, and $|A| \gg 1$ or $|B| \gg 1$. The case $|A| \sim |B|$ can again happen only for $\mathcal{O}(1)$ number of λ's and so we can assume that either $|A| \gg |B|$ or $|B| \gg |A|$. Both cases can be treated equally and so we can assume without loss of generality that $|A| \gg |B|$. Then we can rewrite the phase in the form

$$\lambda \tilde{\Phi}_3(z, s_0, x, \delta, \sigma) = B_0(\lambda, 2^l, z) - B_1(\lambda, 2^l, z) s_0 + b_3(\lambda^{-1/3} s_0, \rho) s_0^3,$$

where we know that for l sufficiently large $|B_0| \sim |A|$, $|B_1| \sim 2^{2l/3}$, and $|b_3| \sim 1$.

In order to simplify the situation a bit, we develop the amplitude function g_3 into a sum of tensor products, separating the s_3 variable from the others. It is sufficient to consider each of these tensor product terms separately, and so we can assume without loss of generality that

$$g_3\Big(2^l, (2^l \lambda^{-1})^{1/3}, z, \lambda^{-1/3} s_0, s_3, \delta, \sigma\Big) = \tilde{g}_3\Big(2^l, (2^l \lambda^{-1})^{1/3}, z, \lambda^{-1/3} s_0, \delta, \sigma\Big)\, \chi_1(s_3),$$

where $\tilde{g}_3$ has the same properties as g_3, except it does not depend on s_3.

Then, after using the Fourier transform in s_3, the integral in s_0 for the function $\tilde{\nu}^E_{\lambda,l}$ is of the form

$$\int \check{\chi}_1\Big(B_0 - B_1 s_0 + b_3(\lambda^{-1/3} s_0, \rho) s_0^3\Big)\, \tilde{g}_3\Big(2^l, (2^l \lambda^{-1})^{1/3}, z, \lambda^{-1/3} s_0, \delta, \sigma\Big) \mathrm{d}s_0, \tag{3.4.13}$$

where we have suppressed the variables of B_0 and B_1. One can easily check that this integral is bounded by $2^{l/3}$ by considering the situations where $|s_0| \lesssim 2^{l/3}$ and $|s_0| \gg 2^{l/3}$ separately. This is in fact true if we use any $L^1 \cap L^\infty$ function instead of $\check{\chi}_1$.

If now $|B_0 - B_1 s_0 + b_3(\lambda^{-1/3}s_0, \rho)s_0^3| \gtrsim |A|^\varepsilon$, by using the Schwartz property we obtain the bound

$$\|\tilde{\nu}_{\lambda,l}^E\|_{L^\infty} \lesssim \lambda^{7/6}2^{-lN}|A|^{-\varepsilon},$$

with a different N, which after plugging into (3.4.10) is summable.

Next, if $|B_0 - B_1 s_0 + b_3(\lambda^{-1/3}s_0, \rho)s_0^3| \ll |A|^\varepsilon$, then

$$B_1 s_0 - b_3(\lambda^{-1/3}s_0, \rho)s_0^3 \in [B_0 - c|A|^\varepsilon, B_0 + c|A|^\varepsilon],$$

for some small $c > 0$. In particular, the fact $|B_0| \sim A$ gives us

$$|B_1 s_0 - b_3(\lambda^{-1/3}s_0, \rho)s_0^3| \sim |A|.$$

First we consider integration over the domain $|s_0| \lesssim 2^{l/3}$. In this case we get

$$|B_1 s_0 - b_3(\lambda^{-1/3}s_0, \rho)s_0^3| \lesssim 2^l,$$

which in turn implies that $|A| \lesssim 2^l$. But this means we can trade a 2^{-l} factor for a $|A|^{-1}$ and so we are done. The second part of the integral is where $|s_0| \gg 2^{l/3}$, which implies $|B_1 s_0 - b_3(\lambda^{-1/3}s_0, \rho)s_0^3| \sim |s_0|^3$, i.e., $|s_0| \sim |A|^{1/3}$. But as the derivative of

$$B_1 s_0 - b_3(\lambda^{-1/3}s_0, \rho)s_0^3$$

is of size $|s_0|^2 \sim |A|^{2/3}$, then if we substitute $t = B_1 s_0 - b_3(\lambda^{-1/3}s_0, \rho)s_0^3$ in the integral (3.4.13), the Jacobian is of size $|A|^{-2/3}$ and so the same $|A|^{-2/3}$ bound holds for the integral. We are done with the estimate for the function $\nu_{\lambda,l}^E$.

3.4.5 Estimates away from the Airy cone – the estimate for $\nu_{\lambda,l}^a$

Again substituting first s for ξ, then s_1 for z, and then s_0 for $s_2^{1/(n-2)}$, we obtain the expression

$$\begin{aligned}\nu_{\lambda,l}^a(x) =&\lambda^{3/2}2^{l/2}\int e^{-i\lambda s_3\Phi_4(z,s_0,x,\delta,\sigma)}\\&\times g_4\Big((2^l\lambda^{-1})^{1/3}, z, s_0, s_3, \delta, \sigma; 2^l\Big)\mathrm{d}z\mathrm{d}s_0\mathrm{d}s_3,\end{aligned}$$

where g_4 is smooth in all of its variables and a classical symbol of order 0 in the last 2^l variable, and where

$$\begin{aligned}\Phi_4(z, s_0, x, \delta, \sigma) :=& s_0^n G_5(s_0^{n-2}, \delta, \sigma) - s_0^{n-1}G_3(s_0^{n-2}, \delta, \sigma)x_1 - s_0^{n-2}x_2 - x_3\\&+ (2^l\lambda^{-1})^{2/3}z(x_1 - s_0 G_1(s_0^{n-2}, \delta, \sigma))\\&+ (2^l\lambda^{-1})z^{3/2}q_0((2^l\lambda^{-1})^{1/3}z^{1/2}, s_0).\end{aligned}$$

We assume $z \sim 1$ since the case $z \sim -1$ can be treated in the same way.

We can restrict ourselves to the case $|x| \lesssim 1$ arguing in the same way as in the previous case. In fact, we can restrict ourselves to the case $|x_1 - s_0 G_1(s_0^{n-2}, \delta, \sigma)| \ll 1$, since otherwise we can use integration by parts in z. From this it follows $|x_1| \sim 1$. Since $G_1(s_0^{n-2}, 0, \sigma) = 1$, we can also localize the integration in s_0 to an arbitrarily small interval containing x_1.

Lemma 3.4.4. *Define the polynomial*

$$P(s_0; x_1, x_2, \sigma) := \frac{(n-1)(n-2)}{2}\sigma\beta(0)s_0^n - n(n-2)\sigma\beta(0)x_1 s_0^{n-1} - x_2 s_0^{n-2}.$$

If $|x_1| \sim \sigma \sim |\beta(0)| \sim 1$*,* $n \geqslant 5$*, and* $|x_2| \lesssim 1$*, then*

$$(n-3)P'(x_1; x_1, x_2, \sigma) = x_1 P''(x_1; x_1, x_2, \sigma),$$

and this expression is a polynomial in (x_1, x_2)*.*

Proof. Factoring out $(n-2)\sigma\beta(0)/2$ we can assume without loss of generality

$$P(s_0; x_1, x_2, \sigma) := (n-1)s_0^n - 2nx_1 s_0^{n-1} - \tilde{x}_2 s_0^{n-2},$$

where $\tilde{x}_2 = (2x_2)/[(n-2)\sigma\beta(0)]$. The first two derivatives of this polynomial are

$$\begin{aligned}
P'(s_0; x_1, x_2, \sigma) &= n(n-1)s_0^{n-1} - 2n(n-1)x_1 s_0^{n-2} - (n-2)\tilde{x}_2 s_0^{n-3}, \\
P''(s_0; x_1, x_2, \sigma) &= n(n-1)^2 s_0^{n-2} - 2n(n-1)(n-2)x_1 s_0^{n-3} - (n-2)(n-3)\tilde{x}_2 s_0^{n-4}.
\end{aligned}$$

Plugging in x_1 we get

$$\begin{aligned}
P'(x_1; x_1, x_2, \sigma) &= -n(n-1)x_1^{n-1} - (n-2)\tilde{x}_2 x_1^{n-3}, \\
P''(x_1; x_1, x_2, \sigma) &= -n(n-1)(n-3)x_1^{n-2} - (n-2)(n-3)\tilde{x}_2 x_1^{n-4},
\end{aligned}$$

and the claim follows. □

The coefficients of the polynomial in the above lemma come from the first three terms of $\Phi_4(z, s_0, x, 0, \sigma)$ and from Lemma 3.4.1. Hence, the above lemma relates the first and the second s_0 derivative of Φ_4 at x_1.

We develop the phase Φ_4 in the variable $u := x_1 - s_0 G_1(s_0^{n-2}, \delta, \sigma)$, which is just a translation of s_0 to x_1 when $\delta = 0$. Then we can write

$$\begin{aligned}
\Phi_4(z, s_0, x, \delta, \sigma) =& b_0(x, \delta, \sigma) + b_1(x, \delta, \sigma)u + b_2(x, \delta, \sigma)u^2 + b_3(x, \delta, \sigma, u)u^3 \\
&+ (2^l\lambda^{-1})^{2/3} z u \\
&+ (2^l\lambda^{-1}) z^{3/2} q_1((2^l\lambda^{-1})^{1/3} z^{1/2}, u),
\end{aligned}$$

where $|q_1| \sim 1$. From Lemma 3.4.4 one easily sees that we can conclude that either $|b_1| \sim |b_2| \sim 1$ or $|b_1|, |b_2| \ll 1$. Since $|u| \ll 1$, the case $|b_1| \sim |b_2| \sim 1$ would imply that we can integrate by parts in u and obtain a factor λ^{-N}. Therefore, we may and shall assume that both $|b_1|$ and $|b_2|$ are very small, and so we can apply Lemma 3.4.2 to obtain

$|b_3| \sim 1$ (this reduction one could have also gotten by checking the third derivative in Lemma 3.4.4).

Now note that if $|u|$ is not of size $(2^l\lambda^{-1})^{1/3}$, then we can apply integration by parts in z to gain a factor 2^{-lN}. In fact, after we substitute $u = (2^l\lambda^{-1})^{1/3}v$, we can get a factor of size $2^{-lN}(1+|v|^2)^{-N/2}$ by integrating by parts in z. Thus, we may restrict ourselves to the discussion of

$$\begin{aligned}
\nu_I^a(x) = \lambda^{7/6}2^{-lN} \int & e^{-i\lambda s_3 \Phi_5(z,v,x,\delta,\sigma)}\,(1+|v|^2)^{-N/2}(1-\chi_1(v))\chi_0((2^l\lambda^{-1})^{1/3}v) \\
& \times \tilde{g}_5\Big((2^l\lambda^{-1})^{1/3}, z, (2^l\lambda^{-1})^{1/3}v, s_3, \delta, \sigma; 2^l\Big)\mathrm{d}z\mathrm{d}v\mathrm{d}s_3, \\
\nu_{II}^a(x) = \lambda^{7/6}2^{5l/6} \int & e^{-i\lambda s_3 \Phi_5(z,v,x,\delta,\sigma)}\chi_1(v) \\
& \times g_5\Big((2^l\lambda^{-1})^{1/3}, z, (2^l\lambda^{-1})^{1/3}v, s_3, \delta, \sigma; 2^l\Big)\mathrm{d}z\mathrm{d}v\mathrm{d}s_3,
\end{aligned}$$

where both g_5 and $\tilde{g}_5$ have the same properties as g_4. In the expression for ν_I^a the $\chi_0((2^l\lambda^{-1})^{1/3}v)$ factor localizes so that $|u| = |(2^l\lambda^{-1})^{1/3}v| \ll 1$. Suppressing dependence on (x, δ, σ), the phase is of the form

$$\begin{aligned}
\lambda\Phi_5&(z,v,x,\delta,\sigma) \\
&= \lambda b_0 + \lambda^{2/3}2^{l/3}b_1 v + \lambda^{1/3}2^{2l/3}b_2 v^2 \\
&+ 2^l\Big(b_3((2^l\lambda^{-1})^{1/3}v)v^3 + zv + z^{3/2}q_1((2^l\lambda^{-1})^{1/3}z^{1/2}, (2^l\lambda^{-1})^{1/3}v)\Big).
\end{aligned} \tag{3.4.14}$$

Estimates for ν_I^a. In this case we plan to use the oscillatory sum lemma in λ only and consider 2^l as a parameter. Let us denote

$$A := \lambda b_0, \qquad B := \lambda^{2/3}2^{l/3}b_1, \qquad D := \lambda^{1/3}2^{2l/3}b_2.$$

We need to reduce our problem to the case when A, B, and D are bounded. As here the integral itself is bounded by $\lesssim 1$, we can assume that it is not the case that $|A| \sim |B|$, nor $|B| \sim |C|$, nor $|A| \sim |C|$, since otherwise λ's would go over a finite set, and we could sum in l. Furthermore, as soon as $|A|$ (resp. $|B|$, or $|C|$) is greater than 1, then we can automatically assume that $|A| \gg 2^{4l}$ (resp. $|B| \gg 2^{4l}$, or $|C| \gg 2^{4l}$), since otherwise we could trade some factors 2^{-lN} to obtain a factor $|A|^{-\varepsilon}$ (resp. $|B|^{-\varepsilon}$, or $|D|^{-\varepsilon}$) giving summability in λ in the expression (3.4.10).

If at least one of $|A|$, $|B|$, or $|C|$ are greater than 1, we define

$$f(v, z, 2^l\lambda^{-1}) := b_3((2^l\lambda^{-1})^{1/3}v)v^3 + zv + z^{3/2}q_1((2^l\lambda^{-1})^{1/3}z^{1/2}, (2^l\lambda^{-1})^{1/3}v),$$

and develop the function $\tilde{g}_5$ into a series of tensor products with variable s_3 separated, i.e., into a sum with terms of the form

$$h((2^l\lambda^{-1})^{1/3}, z, (2^l\lambda^{-1})^{1/3}v, \delta, \sigma; 2^l)\,\chi_1(s_3),$$

where h has the same properties as $\tilde{g}_5$, except it does not depend on s_3. Then after taking the Fourier transform in s_3, we are reduced to estimating the integral

$$\begin{aligned}\lambda^{7/6}2^{-lN}\int&(1+|v|^2)^{-N/2}\chi_1(z)(1-\chi_1(v))\chi_0((2^l\lambda^{-1})^{1/3}v)\\&\times\check{\chi}_1(\lambda b_0+\lambda^{2/3}2^{l/3}b_1v+\lambda^{1/3}2^{2l/3}b_2v^2+2^lf(v,z,2^l\lambda^{-1}))\\&\times h((2^l\lambda^{-1})^{1/3},z,(2^l\lambda^{-1})^{1/3}v,\delta,\sigma;2^l)\mathrm{d}z\mathrm{d}v.\end{aligned}\tag{3.4.15}$$

Case $|v|\ll 1$**.** The bound $|v|\ll 1$ gives

$$\begin{aligned}|f(v,z,2^l\lambda^{-1})|&\sim 1,\\|\partial_vf(v,z,2^l\lambda^{-1})|&\sim 1,\\|\partial_v^2f(v,z,2^l\lambda^{-1})|&\ll 1.\end{aligned}$$

If $|A|\gg\max\{2^{4l},|B|,|D|\}$, then we can easily gain a factor $|A|^{-1}$ using the Schwartz property of $\check{\chi}_1$. If $|B|\gg\max\{2^{4l},|A|,|D|\}$, then the size of the derivative in v of the function within $\check{\chi}_1$ is B and so we get the bound $|B|^{-1}$ by substitution. Finally, if $|D|\gg\max\{2^{4l},|A|,|B|\}$, we use the van der Corput lemma and obtain the bound $|D|^{-1/2}$.

Case $1\ll|v|\ll(2^l\lambda^{-1})^{-1/3}$**.** In this case we can rewrite

$$f(v,z,2^l\lambda^{-1})=v^3\tilde{f}(v,z,(2^l\lambda^{-1})^{-1/3}),$$

where $\tilde{f}$ is a smooth function with $|\tilde{f}|\sim 1$ and $|\partial_v^k\tilde{f}|\ll|v|^{-k}$ for all $k\geqslant 1$. This means that f is behaving essentially like v^3, and in particular

$$\begin{aligned}|f(v,z,2^l\lambda^{-1})|&\sim|v|^3,\\|\partial_vf(v,z,2^l\lambda^{-1})|&\sim|v|^2.\end{aligned}$$

Subcase $\max\{|B|,|D|\}\geqslant 1$**.** As mentioned before, this actually implies that we can assume $\max\{|B|,|D|\}\geqslant 2^{4l}$. If now $|D|\gg|B|$, then since we could otherwise use the factor $(1+|v|^2)^{-N/2}$ in (3.4.15), we can restrain the integration to the domain $|v|\ll|D|^\varepsilon$. Here the derivative in v of the expression

$$A+Bv+Dv^2+2^l\tilde{f}(v,z,2^l\lambda^{-1})v^3\tag{3.4.16}$$

inside the Schwartz function $\check{\chi}_1$ in (3.4.15) is of size $|B+cDv|$ for some $|c|=|c(v)|\sim 1$. But recall that $|v|\gg 1$ and so $|B+cDv|\sim|Dv|\gg|D|$. This means that substituting the above expression would give a Jacobian of size at most $|D|^{-1}$.

Next let us consider the case $|D|\lesssim|B|$. If have the slightly stronger estimate $|D|\lesssim|B|^{1-\varepsilon}$, and if we assume $|v|\ll|B|^\varepsilon$ (which we can because of the factor $(1+|v|^2)^{-N/2}$), then in this case the derivative of (3.4.16) is of size $|B|$, which means substituting this expression yields an admissible bound.

Therefore, we may now consider the case $|B|^{1-\varepsilon}\ll|D|\lesssim|B|$ and $|v|\ll|D|^\varepsilon$, which implies, in case when ε is sufficiently small, that $|D|\geqslant 2^{3l}$. In particular, the derivative of (3.4.16) can be again written as $|B+cDv|$ with $|c|\sim 1$, and we can reduce our problem to the part where $|B+cDv|\ll|D|^\varepsilon$, since otherwise substituting would give a Jacobian of size at most $|D|^{-\varepsilon}$. But now $|B + cDv| \ll |D|^{\varepsilon}$ implies $|v| \sim |B||D|^{-1}$. Hence, it suffices to estimate the integral

$$\int \left|\check{\chi}_1(A + Bv + Dv^2 + 2^l \tilde{f}(v, z, 2^l\lambda^{-1})v^3)\chi_1(|B|^{-1}|D|v)\right| \mathrm{d}v.$$

We substitute $w = |B|^{-1}\,|D|v$ and write

$$|B||D|^{-1} \int \Big|\check{\chi}_1(A + (B|B||D|^{-1})w + (D|B|^2|D|^{-2})w^2 \\ + 2^l|B|^3|D|^{-3}w^3r(w))\chi_1(w)\Big|\mathrm{d}w.$$

Applying the van der Corput lemma we obtain the estimate

$$(|B||D|^{-1})\,(|B|^2|D|^{-1})^{-1/2} = |D|^{-1/2},$$

and so we are done with the case $\max\{|B|, |D|\} \geqslant 1$.

Subcase $\max\{|B|, |D|\} \leqslant 1$ **and** $|A| \gg 1$**.** Again, we may actually assume $|A| \gg 2^{4l}$. We may also then reduce ourselves to the discussion of the case $|v| \ll |A|^{\varepsilon}$, since in the other part of the integration domain we can gain a factor $|A|^{-\varepsilon}$. But then the expression (3.4.16) is of size $\sim |A|$ and we can get a factor $|A|^{-1}$, and hence we are also done with the function ν_I^a.

Estimates for ν_{II}^a. Here we have a non-degenerate critical point in z which would give us a factor $2^{-l/2}$. We shall not apply directly the stationary phase method here since in this case some crucial information has been lost while we were deriving the form of the phase in this and the previous subsections. It seems that one cannot prove the required bound for complex interpolation using the information from the form of the phase (3.4.14). One needs to go back to the phase form in the original coordinates (the one before taking the inverse Fourier transform is (3.4.1)) and find the critical point in the variables (y_1, s_1). This was carried out in [51] (see the discussion before [51, Lemma 5.6.]). Here we only sketch the steps.

The phase in (3.4.1) is

$$\Psi(y_1, \delta, \sigma, s_1, s_2) = s_1y_1 + s_2y_1^2\omega(\delta_1y_1) + \sigma y_1^n\beta(\delta_1y_1) + (\delta_0s_2)^2Y_3(\delta_1y_1, \delta_2, \delta_0s_2),$$

and one integrates in the y_1 variable. The phase function after one applies the Fourier transform is

$$\Phi_0(y_1, s_1, s_2, x, \delta, \sigma) = \Psi(y_1, \delta, \sigma, s_1, s_2) - s_1x_1 - s_2x_2 - x_3, \tag{3.4.17}$$

and one now integrates in the s and y_1 variables, after substituting s for ξ. Recall that $s_0 = s_2^{1/(n-2)}$ and

$$\begin{aligned} s_1 &= s_0^{n-1}G_3(s_0^{n-2}, \delta, \sigma) - \lambda^{-2/3}z, \\ v &= (2^l\lambda^{-1})^{-1/3}(x_1 - s_0G_1(s_0^{n-2}, \delta, \sigma)). \end{aligned}$$

Therefore fixing (s_2, s_3) is equivalent to fixing (v, s_3), and in this case, finding the critical point in (y_1, s_1) is equivalent to finding the critical point in the (y_1, z) coordinates. Recall that the phase form in (3.4.14) was derived by using the stationary phase method in y_1 (implicitly done as a part of Lemma 2.2.2) and changing variables from $s = (s_1, s_2, s_3)$ to (z, v, s_3).

The key is now to notice that since the critical point is invariant with respect to coordinate changes, and so, after applying the stationary phase in z to the phase function (3.4.14), we get

$$\Phi_5(z^c, v, x, \delta, \sigma),$$

which is the equal to the phase function in (3.4.17) after we apply the stationary phase in (y_1, s_1):

$$\Phi_0(y_1^c, s_1^c, s_2, x, \delta, \sigma),$$

and then change the coordinates from s_2 to v. This was carried out in [51] by explicitly calculating the critical point in (y_1, s_1) in (3.4.17) (see [51, Lemma 5.6]). One obtains that we can rewrite the function ν_{II}^a as

$$\nu_{II}^a(x) = \lambda^{7/6} 2^{l/3} \int e^{-i\lambda s_3 \Phi_6(\tilde{v}, x, \delta, \sigma)} g_6\Big((2^l\lambda^{-1})^{1/3}, \tilde{v}, s_3, \delta, \sigma; 2^l\Big) \chi_1(\tilde{v}) \mathrm{d}\tilde{v} \mathrm{d}s_3,$$

where

$$\begin{aligned} \lambda\Phi_6(\tilde{v}, x, \delta, \sigma) =& \lambda\tilde{b}_0(x, \delta, \sigma) + \lambda^{2/3} 2^{l/3} \tilde{b}_1(x, \delta, \sigma)\tilde{v} \\ &+ \delta_0^2 2^{2l/3} \lambda^{1/3} \tilde{b}_2(x, \delta_0 (2^l\lambda^{-1})^{1/3}\tilde{v}, \delta, \sigma)\tilde{v}^2, \end{aligned}$$

with $\tilde{b}_0$, $\tilde{b}_1$, $\tilde{b}_2$ smooth, and $|\tilde{b}_2| \sim 1$. The amplitude g_6 is a classical symbol of order 0 in 2^l, but we shall ignore this dependence since the lower order terms can be treated similarly, and even simpler since we can gain summability in l and use the one parameter oscillatory sum lemma for λ.

We remark that the variable $\tilde{v}$ is only slightly different from the variable v defined above after the statement of Lemma 3.4.4. Here $\tilde{v}$ corresponds to the v variable of [51, Subsection 5.2.3]. We explain briefly the relation between v and $\tilde{v}$. At the beginning of this subsection we obtained ν_{II}^a by localising to the part where

$$|(2^l\lambda^{-1})^{1/3} v| = |x_1 - s_0 G_1(s_0^{n-2}, \delta, \sigma)| = |x_1 - s_2^{n-2} G_1(s_2, \delta, \sigma)| \sim (2^l\lambda^{-1})^{1/3},$$

i.e., $|v| \sim 1$. Since $G_1(s_2, 0, \sigma) = 1$, one can easily see by using the implicit function theorem that solving the equation

$$x_1 - s_2^{n-2} G_1(s_2, \delta, \sigma) = (2^l\lambda^{-1})^{1/3} v$$

in s_2 one can write

$$s_2 = \tilde{G}_1(x_1, \delta, \sigma) + (2^l\lambda^{-1})^{1/3} v\, \tilde{G}((2^l\lambda^{-1})^{1/3} v, x_1, \delta, \sigma),$$

where $|\tilde{G}| \sim \tilde{G}_1 \sim 1$. Therefore if the $\tilde{v}$ variable is defined by

$$\tilde{v} = (2^l\lambda^{-1})^{-1/3} (s_2 - \tilde{G}_1(x_1, \delta, \sigma)),$$

as is v of [51], then

$$\tilde{v} = v\tilde{G}((2^l\lambda^{-1})^{1/3}v, x_1, \delta, \sigma).$$

In particular, there is no significant difference between v and $\tilde{v}$.

We define

$$A := \lambda\tilde{b}_0(x,\delta,\sigma), \qquad B := \lambda^{2/3}2^{l/3}\tilde{b}_1(x,\delta,\sigma), \qquad D := \delta_0^2 2^{2l/3}\lambda^{1/3},$$

suppress the variables of $\tilde{b}_2$, and shorten $\rho = \delta_0(2^l\lambda^{-1})^{1/3}$. Then

$$\lambda\Phi_6(\tilde{v}, x, \delta, \sigma) = A + B\tilde{v} + D\tilde{b}_2(\rho\tilde{v})\tilde{v}^2,$$

and in order to use the oscillatory sum lemma for two parameters we need to reduce the problem to the situation where $|A|$, $|B|$, and $|D|$ are of size $\lesssim 1$. In the following we define k through $\lambda = 2^k$.

First we treat the case when at least two of $|A|$, $|B|$, and $|D|$ are comparable. When this is the case, λ can go over only a finite set of indices (the index sets depending on l and other constants), and it remains to sum only in l. This is done in the following way. If $|D| \gtrsim 1$, then we can use van der Corput lemma and obtain a factor $|D|^{-1/2}$, which is summable in l. If $|D| \ll 1$, then the only case remaining is $|A| \sim |B|$, and here we can use integration by parts in $\tilde{v}$ and obtain a factor $|B|^{-1}$ which we use to sum in l.

Next, we assume that we have a "strict order" between $|A|$, $|B|$, and $|D|$. First we shall consider the cases when at least two of $|A|$, $|B|$, and $|D|$ are greater than 1. If $|A| \gg \max\{|B|, |D|\} \gtrsim 1$, we use integration by parts in s_3 and obtain

$$|A|^{-1} \ll |A|^{-1/2}|\max\{|B|, |D|\}|^{-1/2},$$

which is summable. Similarly, if $|B| \gg \max\{|A|, |D|\} \gtrsim 1$, we can integrate by parts in $\tilde{v}$ and obtain the estimate

$$|B|^{-1} \ll |B|^{-1/2}|\max\{|A|, |D|\}|^{-1/2},$$

which is summable. And if now $|D| \gg \max\{|A|, |B|\} \gtrsim 1$, we use the van der Corput lemma and obtain

$$|D|^{-1/2} \ll |D|^{-1/4}|\max\{|A|, |B|\}|^{-1/4},$$

which is again summable. We are thus reduced to the case where one of $|A|$, $|B|$, or $|D|$ are greater than 1, and the other two much smaller.

Case $|A| \geqslant 1$ and $\max\{|B|, |D|\} \ll 1$. In this case by using integration by parts in s_3 we can get a factor $|A|^{-1}$. We use the one dimensional oscillatory sum lemma in l, and afterwards, we can sum in λ using the factor $|A|^{-1}$ which can be obtained as the bound on the C^1 norm of the function to which we applied the oscillatory sum lemma.

Case $|B| \geqslant 1$ and $\max\{|A|, |D|\} \ll 1$. Here we change the summation variables

$$2^{k_1} := \lambda^2 2^l,$$
$$2^{k_2} := \lambda,$$

so that we now sum over (k_1, k_2). This change of variables corresponds to the system

$$\begin{aligned} k_1 &= 2k + l, \\ k_2 &= k, \end{aligned}$$

which has determinant equal to 1, and so the associated linear mapping is a bijection on $\mathbb{Z}^2$.

Since the summation bounds (without the constraints set by A, B, or D) are $1 \ll \lambda \leqslant \delta_0^{-6}$ and $1 \ll 2^l \ll \lambda$, for each fixed k_1 the summation in k_2 is now within the range $2^{k_1/3} \ll 2^{k_2} \ll 2^{k_1/2}$, and the summation in k_1 is for $1 \ll 2^{k_1} \ll \delta_0^{-18}$.

The quantities B and D can be rewritten as

$$\begin{aligned} B &= 2^{k_1/3}\tilde{b}_1(x, \delta, \sigma), \\ D &= \delta_0^2\, 2^{2k_1/3 - k_2}. \end{aligned}$$

Now for a fixed k_1 we can apply the one-dimensional oscillatory sum lemma to sum in 2^{k_2} since all the terms coupled with 2^{k_2} are now within a bounded range. In order to sum in k_1, one needs to estimate the C^1 norm of the function to which we have applied the oscillatory sum lemma. One can easily see that integrating by parts in s_0 we obtain a factor $|B|^{-1}$ which in the new indices depends only on 2^{k_1}.

Case $|D| \geqslant 1$ and $\max\{|A|, |B|\} \ll 1$. In this case we also change the summation variables

$$\begin{aligned} 2^{k_1} &:= \lambda 2^{2l}, \\ 2^{k_2} &:= \lambda, \end{aligned}$$

so that we now sum over (k_1, k_2). We have

$$\begin{aligned} k_1 &= k + 2l, \\ k_2 &= k, \end{aligned} \qquad \Longleftrightarrow \qquad \begin{aligned} k &= k_2, \\ l &= (k_1 - k_2)/2. \end{aligned}$$

Therefore when we fix k_1, the summation in k_2 goes over an interval of even or uneven integers, depending on the parity of k_1. Since the summation bounds (without the constraints set by A, B, or D) are $1 \ll \lambda \leqslant \delta_0^{-6}$ and $1 \ll 2^l \ll \lambda$, for each k_1 the summation in k_2 is now within the range $2^{k_1/3} \ll 2^{k_2} \ll 2^{k_1}$, and the summation in k_1 is for $1 \ll 2^{k_1} \ll \delta_0^{-18}$.

The quantities B and D can be rewritten as

$$\begin{aligned} B &= 2^{k_1/2 + 3k_2/2}\tilde{b}_1(x, \delta, \sigma), \\ D &= \delta_0^2\, 2^{k_1/3}. \end{aligned}$$

For a fixed k_1 we want to apply the oscillatory sum lemma to the summation in k_2. We remark that formally one should write k_2 as either $2r + 1$ or $2r$ (depending on the parity of k_1), and then apply the oscillatory sum lemma to the summation in r instead of k_2.

Here we give a bit more details compared to the previous case since the term ρ, which contains $(2^l\lambda^{-1})^{1/3}$, is coupled with D. We need to estimate the C^1 norm of the function

$$H(z_1, z_2, z_3; x, \delta, \sigma) := \int e^{-is_3(z_1 + z_2\tilde{v} + D\tilde{b}_2(x, z_3\delta_0\tilde{v}, \delta, \sigma)\tilde{v}^2)} g_6(z_3, \tilde{v}, s_3, \delta, \sigma)\chi_1(\tilde{v})\mathrm{d}\tilde{v}\mathrm{d}s_3.$$

Formally, one should also add further dummy z_i's for controlling the range of the summation indices. Since we are in the case where $|D| \geqslant 1$, $|z_1| \ll 1$, and $|z_2| \ll 1$, integrating by parts in s_3 we get that the L^∞ estimate is $|D|^{-1}$. Taking derivatives in z_1 and z_2 does not change the form of the integral in an essential way, and so we can also estimate the L^∞ norm of the these derivatives by $|D|^{-1}$. Taking the derivative in z_3 a factor of size at most $|D|$ appears, but now we just apply integration by parts in s_3 two times and get that we can estimate the C^1 norm of H by $|D|^{-1}$.

Case $|A| \lesssim 1$, $|B| \lesssim 1$, and $|D| \lesssim 1$. Here we apply the two-parameter oscillatory sum lemma. We only need to check the additional linear independence condition appearing in the assumptions of Lemma 2.2.7. The terms where $\lambda = 2^k$ and 2^l appear are

$$A = 2^{\beta_1^1 k}\tilde{b}_0(x, \delta, \sigma), \qquad B = 2^{\beta_1^2 k + \beta_2^2 l} 2^{l/3} \tilde{b}_1(x, \delta, \sigma),$$
$$D = \delta_0^2 \, 2^{\beta_1^3 k + \beta_2^3 l}, \qquad (2^l \lambda^{-1})^{1/3} = 2^{\beta_1^4 k + \beta_2^4 l},$$

where

$$(\beta_1^1, \beta_2^1) = (1, 0), \qquad (\beta_1^2, \beta_2^2) = (2/3, 1/3),$$
$$(\beta_1^3, \beta_2^3) = (1/3, 2/3), \qquad (\beta_1^4, \beta_2^4) = (-1/3, 1/3),$$

and recall from (3.4.10) that

$$(\alpha_1, \alpha_2) = (-7/4, -1/2).$$

Formally, we also have to consider additionally

$$(\beta_1^5, \beta_2^5) = (-1, 0), \qquad (\beta_1^6, \beta_2^6) = (0, -1),$$

for implementing the lower summation bounds for λ and 2^l as in (3.4.10). We see that the condition $\alpha_1 \beta_2^r \neq \alpha_2 \beta_1^r$ is satisfied for each $r = 1, \ldots, 6$. Therefore, we may now apply the lemma and obtain the inequality (3.4.10). This finishes the proof of Theorem 3.3.1.

Chapter 4

Fourier restriction for mixed homogeneous surfaces

As announced in Section 1.4, in this chapter we prove Fourier restriction estimates for surfaces given as graphs of smooth functions $\phi : \mathbb{R}^2 \backslash \{0\} \to \mathbb{R}$ which are κ-mixed homogeneous of degree $\mathcal{D}$:

$$\phi(r^{\kappa_1} x_1, r^{\kappa_2} x_2) = r^{\mathcal{D}} \phi(x_1, x_2), \qquad r > 0, \tag{4.0.1}$$

where $\kappa \in (0,\infty)^2$ and $\mathcal{D} \in \{-1, 0, 1\}$. Both κ and $\mathcal{D}$ shall be fixed throughout this chapter. In particular, we are interested in the estimate

$$\|\widehat{f}\|_{L^2(\mathrm{d}\mu)} \leqslant C \|f\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})}, \qquad f \in \mathcal{S}(\mathbb{R}^3), \tag{4.0.2}$$

where μ is defined by

$$\langle \mu, f \rangle = \int_{\mathbb{R}^2 \backslash \{0\}} f(x_1, x_2, \phi(x_1, x_2)) \, \mathcal{W}(x_1, x_2) \, \mathrm{d}x. \tag{4.0.3}$$

We remind that the weight $\mathcal{W}$ is κ-mixed homogeneous of degree $\mathcal{D}_{\mathcal{W}}$ and that we consider two particular cases, namely, the case when $\mathcal{W}$ is either of the form

$$|x|_\kappa^{\mathcal{D}_{\mathcal{W}}} = (|x_1|^{1/\kappa_1} + |x_2|^{1/\kappa_2})^{\mathcal{D}_{\mathcal{W}}},$$

or of the form

$$|\mathcal{H}_\phi(x)|^{\mathfrak{s}} = \left| \det \begin{bmatrix} \partial_{x_1}^2 \phi & \partial_{x_1} \partial_{x_2} \phi \\ \partial_{x_1} \partial_{x_2} \phi & \partial_{x_2}^2 \phi \end{bmatrix} \right|^{\mathfrak{s}},$$

where $\mathcal{D}_{\mathcal{W}} = 2\mathfrak{s}(\mathcal{D} - |\kappa|)$. In order to achieve scaling invariance for μ, the quantity $\mathcal{D}_{\mathcal{W}}$ has to satisfy a certain relation whcih we determine in Subsection 4.1.1 (and in particular in Proposition 4.1.1).

This chapter is structured in the following way. In Section 4.1 we first perform some elementary reductions. Since the proofs of Theorem 1.4.1 and Theorem 1.4.2 are essentially based on Proposition 1.4.4, we first prove this proposition (and even obtain slightly more precise results) in Section 4.2. Subsequently we prove Theorems 1.4.1 and 1.4.2 in the respective Sections 4.3 and 4.4. In the last section we then give a sketch of the proof of Corollary 1.4.5.

We remind that r and q (possibly with subscripts and tildes) are used in this chapter generically as smooth functions which do not vanish at the origin. Sometimes they also represent flat functions, in which case we state this explicitly.

4.1 Preliminary reductions

4.1.1 Rescaling and reduction to local estimates

Recall that in this case we consider the measure

$$\langle \mu, f \rangle = \int_{\mathbb{R}^2 \setminus \{0\}} f(x_1, x_2, \phi(x_1, x_2)) \, \mathcal{W}(x_1, x_2) \, \mathrm{d}x,$$

where $\mathcal{W}$ is nonnegative, continuous on $\mathbb{R}^2 \setminus \{0\}$, and κ-mixed homogeneous of degree $\mathcal{D}_{\mathcal{W}}$. In this subsection we determine the degree of homogeneity $\mathcal{D}_{\mathcal{W}}$ so that the global Fourier restriction estimate (4.0.2) becomes equivalent to the local one. By this we mean the following. Let us take a partition of unity $(\eta_j)_{j \in \mathbb{Z}}$ in $\mathbb{R}^2 \setminus \{0\}$:

$$\sum_{j \in \mathbb{Z}} \eta_j(x) = 1, \qquad x \neq 0, \tag{4.1.1}$$

such that $\eta_j = \eta \circ \delta_{2^{-j}}$ for some $\eta = \eta_0 \in C_c^\infty(\mathbb{R}^2)$ supported away from the origin. Let us consider the measures

$$\langle \mu_j, f \rangle := \int_{\mathbb{R}^2} f(x, \phi(x)) \, \eta_j(x) \, \mathcal{W}(x) \, \mathrm{d}x, \tag{4.1.2}$$

which now satisfy $\mu = \sum_{j \in \mathbb{Z}} \mu_j$, and let us furthermore assume that we have the local estimate for some $j_0 \in \mathbb{Z}$:

$$\|\widehat{f}\|_{L^2(\mathrm{d}\mu_{j_0})} \leqslant C \|f\|_{L^p},$$

where $L^p = L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})$. We want to determine the degree of homogeneity of $\mathcal{W}$ so that the Fourier restriction estimate is invariant under the dilations δ_r, i.e., that we have

$$\|\widehat{f}\|_{L^2(\mathrm{d}\mu_j)} \leqslant C \|f\|_{L^p} \tag{4.1.3}$$

for all $j \in \mathbb{Z}$ whenever the estimate is true for some $j_0 \in \mathbb{Z}$. In this case, and if $(p_1, p_3) \in (1, 2]^2$, a standard Littlewood-Paley argument will then yield

$$\|\widehat{f}\|_{L^2(\mathrm{d}\mu)} \leqslant C \|f\|_{L^p}.$$

To summarize, we have:

Proposition 4.1.1. *Let $\mathcal{W}$ be κ-mixed homogeneous of degree $\mathcal{D}_{\mathcal{W}}$, not identically zero, and continuous on $\mathbb{R}^2\backslash\{0\}$, let μ be defined as in* (4.0.3)*, and let $p_1, p_3 \in (1,2]$. Then the Fourier restriction estimate* (4.0.2) *for μ is equivalent to the Fourier restriction estimate* (4.1.3) *for the measure μ_j for any $j \in \mathbb{Z}$ (as defined in* (4.1.2)*) if and only if*

$$\mathcal{D}_{\mathcal{W}} = 2\Big(\frac{|\kappa|}{p_1'} + \frac{\mathcal{D}}{p_3'} - \frac{|\kappa|}{2}\Big) \tag{4.1.4}$$

is satisfied.

Proof. Let us first determine what $\mathcal{D}_{\mathcal{W}}$, the degree of homogeneity of $\mathcal{W}$, needs to be in order that (4.1.3) holds true for all $j \in \mathbb{Z}$ whenever the it holds true for some $j_0 \in \mathbb{Z}$. Recall that $|\delta_r x|_\kappa = r|x|_\kappa$. Inspecting the definition (4.1.2) of μ_j one gets:

$$\langle \mu_j, f\rangle = 2^{j|\kappa| + j\mathcal{D}_{\mathcal{W}}} \langle \mu_0, \mathrm{Dil}_{(2^{-j\kappa_1}, 2^{-j\kappa_2}, 2^{-j\mathcal{D}})} f\rangle,$$

where $(\mathrm{Dil}_{(\lambda_1,\lambda_2,\lambda_3)} f)(x_1,x_2,x_3) = f(\lambda_1^{-1}x_1, \lambda_2^{-1}x_2, \lambda_3^{-1}x_3)$. The above relation can be interpreted as

$$\mu_j = 2^{j\mathcal{D}_{\mathcal{W}} - j\mathcal{D}} \, \mathrm{Dil}_{(2^{j\kappa_1}, 2^{j\kappa_2}, 2^{j\mathcal{D}})} \, \mu_0.$$

Let us assume that we have for some $j \in \mathbb{Z}$ the estimate

$$\langle \mu_j, |\widehat{f}|^2\rangle = \|\widehat{f}\|^2_{L^2(\mathrm{d}\mu_j)} \leqslant C^2 \|f\|^2_{L^p}.$$

Since the Fourier transform behaves well with respect to dilations $\mathrm{Dil}_{(\lambda_1,\lambda_2,\lambda_3)}$, we may rescale the above estimate and get

$$\|\widehat{f}\|_{L^2(\mathrm{d}\mu_0)} \leqslant C 2^{-j|\kappa|/2 - j\mathcal{D}_{\mathcal{W}}/2 + j(\kappa_1/p_1' + \kappa_2/p_1' + \mathcal{D}/p_3')} \|f\|_{L^p}.$$

From this one sees that we need precisely (4.1.4) in order for the constant in (4.1.3) to be independent of j. If (4.1.4) does not hold, then the constant blows up in one of the cases $j \to \infty$ or $j \to -\infty$, and in particular, the Fourier restriction estimate (4.0.2) for μ cannot hold (here we use that the restriction operators for μ and μ_j's are nonzero since $\mathcal{W}$ is not identically zero).

Let us now assume that we indeed have (4.1.4). It is obvious that the Fourier restriction estimate for μ implies the Fourier restriction estimate for μ_j for any j. Let us therefore assume that the estimate (4.1.3) holds true for any $j \in \mathbb{Z}$, and thus for all $j \in \mathbb{Z}$.

Before proceeding further let us denote by $(\tilde{\eta}_j)_{j\in\mathbb{Z}}$ a family of $C_c^\infty(\mathbb{R}^2\backslash\{0\})$ functions such that

$$\tilde{\eta}_j = \tilde{\eta}_0 \circ \delta_{2^{-j}} \qquad \text{for all}\, j \in \mathbb{Z},$$

and such that $\tilde{\eta}_j$ is equal to 1 on the support of η_j. One can for example take $\tilde{\eta}_j = \sum_{|k-j|\leqslant N} \eta_k$ for some sufficiently large N. Let us furthermore denote by S_j the Fourier multiplier operator in $\mathbb{R}^3$ with multiplier $(\tilde{\eta}_j \otimes 1)(\xi_1,\xi_2,\xi_3) = \tilde{\eta}_j(\xi_1,\xi_2)$.

Now (4.1.3) implies

$$\|\widehat{S_j f}\|_{L^2(\mathrm{d}\mu_j)} = \|\widehat{f}\|_{L^2(\mathrm{d}\mu_j)} \leqslant C\|S_j f\|_{L^p}.$$

Therefore

$$\begin{aligned}\|\widehat{f}\|^2_{L^2(\mathrm{d}\mu)} &= \langle \mu, |\widehat{f}|^2\rangle = \sum_{j\in\mathbb{Z}} \Big\langle \mu_j, |\widehat{f}|^2 \Big\rangle = \sum_{j\in\mathbb{Z}} \Big\langle \mu_j, |\widehat{S_j f}|^2 \Big\rangle \\ &\leqslant C^2 \sum_{j\in\mathbb{Z}} \|S_j f\|^2_{L^p} = C^2 \Big\| \|S_j f\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})} \Big\|^2_{l^2_j},\end{aligned}$$

where l^2_j denotes the norm of the Hilbert space of l^2 sequences on $\mathbb{Z}$. Since both $p_1 \leqslant 2$ and $p_3 \leqslant 2$ we may use Minkowski's inequality to interchange the l^2_j norm with the L^p norm, and subsequently apply Littlewood-Paley theory in the (x_1, x_2) variable (in particular, we do not need to use mixed norm Littlewood-Paley theory) to get

$$\Big\| \|S_j f\|_{L^p} \Big\|^2_{l^2_j} \leqslant \Big\| \|S_j f\|_{l^2_j} \Big\|^2_{L^p} = \Big\| \Big(\sum_{j\in\mathbb{Z}} |S_j f|^2 \Big)^{1/2} \Big\|^2_{L^p} \sim \|f\|_{L^p}.$$

This finishes the proof of Proposition 4.1.1. □

Remark 4.1.2 (Scaling in the case of Hessian determinant)**.** *Using the homogeneity condition of ϕ one easily obtains that the Hessian determinant is also κ-mixed homogeneous of degree $2\mathcal{D} - 2|\kappa|$. Thus, when we take $\mathcal{W} = |\mathcal{H}_\phi|^{\mathfrak{s}}$, $\mathcal{W}$ is homogeneous of degree $\mathcal{D}_\mathcal{W} = 2\mathfrak{s}(\mathcal{D} - |\kappa|)$. Recall that in this case (i.e., as in the assumptions of Theorem 1.4.1) we assume that $1/p_1' = 1/2 - \mathfrak{s}$, $1/p_3' = \mathfrak{s}$, and so by* (4.1.4) *the equality $\mathcal{D}_\mathcal{W} = 2\mathfrak{s}(\mathcal{D} - |\kappa|)$ is indeed satisfied, i.e., the right relation between the exponents if one wants scaling invariance.*

Remark 4.1.3 (A general sufficient condition for local integrability of $\mathcal{W}$)**.** *Since $\mathcal{W}$ is mixed homogeneous of degree $\mathcal{D}_\mathcal{W}$, $\mathcal{W}|x|_\kappa^{-\mathcal{D}_\mathcal{W}}$ is mixed homogeneous of degree 0, and in particular a bounded function. Thus $|\mathcal{W}| \lesssim |x|_\kappa^{\mathcal{D}_\mathcal{W}}$, and so it is sufficient to check when $|x|_\kappa^{\mathcal{D}_\mathcal{W}}$ is locally integrable in $\mathbb{R}^2$. By symmetry it is sufficient to integrate over $\{(x_1, x_2) : x_1, x_2 > 0\}$. We have*

$$\begin{aligned}\int_{x_1,x_2>0,|x|\lesssim 1} |x|_\kappa^{\mathcal{D}_\mathcal{W}} \mathrm{d}x &= \int_{x_1,x_2>0,|x|\lesssim 1} (x_1^{1/\kappa_1} + x_2^{1/\kappa_2})^{\mathcal{D}_\mathcal{W}} \mathrm{d}x \\ &\sim \int_{y_1,y_2>0,|y|\lesssim 1} (y_1^2 + y_2^2)^{\mathcal{D}_\mathcal{W}} y_1^{2\kappa_1 - 1} y_2^{2\kappa_2 - 1} \mathrm{d}y \\ &\sim \int_{0<r\lesssim 1} \int_0^{\pi/2} r^{2\mathcal{D}_\mathcal{W} + 2|\kappa| - 1} (\cos\theta)^{2\kappa_1 - 1} (\sin\theta)^{2\kappa_2 - 1} \mathrm{d}\theta \mathrm{d}r\end{aligned}$$

Therefore, we must have $2\mathcal{D}_\mathcal{W} + 2|\kappa| - 1 > -1$, i.e.,

$$\mathcal{D}_\mathcal{W} + |\kappa| > 0.$$

Note that this holds if $\mathcal{D} \geqslant 0$, $p_1 > 1$, and $\mathcal{D}_\mathcal{W}$ is given by (4.1.4)*.*

Remark 4.1.4. *When ϕ is smooth at the origin and a nonconstant function, then $\mathcal{D} = 1$, and the necessary condition obtained by a Knapp-type example associated to the principle face of $\mathcal{N}(\phi)$ in the initial coordinate system (see Proposition 3.1.1) tells us that*

$$\frac{|\kappa|}{p_1'} + \frac{1}{p_3'} \leqslant \frac{|\kappa|}{2}$$

is necessary for (4.0.2) *if* $\mathcal{W} \equiv 1$ *(i.e.,* $\mathcal{D}_{\mathcal{W}} = 0$*). On the other hand, if we denote* $l_\kappa = \{(t_1, t_3) \in \mathbb{R}^2 : |\kappa|t_1 + t_3 = |\kappa|/2\}$*, then the expression* (4.1.4) *for* $\mathcal{D}_{\mathcal{W}}$ *implies that*

$$|\mathcal{D}_{\mathcal{W}}| = 2\sqrt{1 + |\kappa|^2}\ \mathrm{dist}\Big((1/p_1', 1/p_3'), l_\kappa\Big).$$

4.1.2 Some further reductions

According to Proposition 4.1.1, under the conditions of Theorem 1.4.1 or Theorem 1.4.2, we have to prove the Fourier restriction estimate for a measure defined by the mapping

$$f \mapsto \int_{\mathbb{R}^2} f(x, \phi(x))\, \eta(x)\, \mathcal{W}(x) \mathrm{d}x,$$

where $\eta \in C_c^\infty(\mathbb{R}^2 \backslash \{0\})$ is supported in a compact annulus centered at the origin. Note that in the case of the weight $\mathcal{W} = |\mathcal{H}_\phi|^{\mathfrak{s}}$ (the case of Theorem 1.4.1) the degree of homogeneity $\mathcal{D}_{\mathcal{W}} = 2\mathfrak{s}(\mathcal{D} - |\kappa|)$ satisfies the relation (4.1.4) by Remark 4.1.2.

Reductions for the amplitude η. One can easily show that in the context of the Fourier restriction problem we may make the following reductions. First, by reordering coordinates and/or changing their sign, and by splitting the amplitude η into functions with smaller support, we may restrict ourselves to amplitudes η with support contained in the half-plane $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \gtrsim 1\}$. Then, by compactness, we may localize to small neighbourhoods of points $v \neq 0$ having $v_1 \gtrsim 1$. Thus, one may assume that the support of η is contained in a small neighbourhood of some generic point v satisfying $v_1 \sim 1$ and $|v| \lesssim 1$. In fact, compactness and changing signs if necessary implies that we may further assume that either $v_2 = 0$ or $v_2 \sim 1$.

Changing the affine terms of the phase. By the previous discussion it suffices to consider the measure

$$f \mapsto \int_{\mathbb{R}^2} f(x, \phi(x))\, \eta_v(x)\, \mathcal{W}(x) \mathrm{d}x, \tag{4.1.5}$$

where η_v is a smooth function supported in a small neighbourhood of a point $v \neq 0$. We now recall the fact that we can freely add or remove linear and constant terms in the expression for ϕ in the context of the Fourier restriction problem. For the constant term this is obvious. For the linear terms this can be achieved by using a linear transformation of the form $(x_1, x_2, x_3) \mapsto (x_1, x_2, b_1x_1 + b_2x_2 + x_3)$ (for more details see Section 2.1). In particular, instead of considering the measure (4.1.5), we may consider the measure

$$f \mapsto \int_{\mathbb{R}^2} f(x, \phi_v(x - v))\, \eta_v(x)\, \mathcal{W}(x) \mathrm{d}x,$$

where we recall that

$$\phi_v(x) := \phi(x + v) - \phi(v) - x \cdot \nabla\phi(v).$$

The strategy for the proof of Theorem 1.4.1 and Theorem 1.4.2 should now be clear. The above discussion reduces the problem to proving a local Fourier restriction estimate in the vincinity of a point v, and so one needs to determine the local normal form of ϕ at v, and in the case $\mathcal{W}(x) = |\mathcal{H}_\phi(x)|^{\mathfrak{s}}$ one needs to additionally determine the order of vanishing of the Hessian determinant at v in the x_2 direction (after which the normal form of $\mathcal{W}$ will be clear by homogeneity).

4.2 Local normal forms

In this section we derive the local normal forms for ϕ and for the Hessian determinant $\mathcal{H}_\phi$ at a fixed point $v \neq 0$ (as a consequence we prove Proposition 1.4.4). The discussion in Subsection 4.1.2 implies that we may assume that $v_1 \sim 1$, and either $v_2 = 0$ or $v_2 \sim 1$.

The structure of this section is as follows. In Subsection 4.2.1 we fix the notation for this section, introduce relevant quantities, and define the coordinate systems y, z, and w (the coordinate systems z and w will not be described precisely until Subsection 4.2.5 though). In Subsections 4.2.2, 4.2.3, and 4.2.4 tables with normal forms of ϕ_v are given. It turns out that in most cases y coordinates suffice and when we use them one obtains the normal forms easily. We deal with the case when y coordinates do not suffice in Subsection 4.2.5. In Subsection 4.2.6 we sketch how to calculate what is the order of vanishing of the Hessian determinant for the respective normal forms.

We assume that the (H1) condition is satisfied throughout this section, i.e., that at any given point $(x_1, x_2) \neq (0,0)$ where the Hessian determinant of ϕ vanishes at least one of the mappings $t \mapsto \partial_1^2\phi(t, x_2)$ or $t \mapsto \partial_2^2\phi(x_1, t)$ is of finite type at $t = x_1$ (resp. $t = x_2$). In fact, in Subsection 4.2.2 we shall explicitly determine the local normal form of ϕ when $t \mapsto \partial_2^2\phi(v_1, t)$ is flat at v_2. In this case it turns out that the Hessian determinant either does not vanish at v, or that it is flat at v. In all the other subsections we shall assume that $t \mapsto \partial_2^2\phi(v_1, t)$ is of finite type at v_2.

4.2.1 Notation and some general considerations

Let us begin by introducing the notation. It will be useful to denote

$$m := \frac{\kappa_2}{\kappa_1} > 0,$$

and for the point $v = (v_1, v_2)$ (recall $v_1 \sim 1$) we define

$$t_0 := v_2 v_1^{-m}.$$

Note that, unlike in Chapter 3, here m can be any positive real number. Let us denote the ∂_2 derivatives of ϕ at $(1, t_0)$ by

$$b_j := \partial_2^j\phi(1, t_0) = g^{(j)}(t_0), \qquad j \in \mathbb{N}_0,$$

where

$$g(t) := \phi(1, t).$$

We furthermore denote

$$k := \inf\{j \geqslant 2 : b_j \neq 0\}, \tag{4.2.1}$$

where we take $k = \infty$ if $b_j = 0$ for all $j \geqslant 2$. The equality $k = \infty$ is equivalent to $g^{(2)}$ being flat at 0. What precisely happens when $g^{(2)}$ is flat shall be explained in Subsection

4.2.2, and in the rest of the section (including this subsection) we assume that $k < \infty$, unless explicitly stated otherwise.

General form of mixed homogeneous ϕ. Recall that we denote by $\mathcal{D} \in \{-1, 0, 1\}$ the degree of homogeneity of ϕ. Then we have for any x satisfying $x_1 > 0$:

$$\phi(x_1, x_2) = x_1^{\mathcal{D}/\kappa_1} \phi(1, x_2 \, x_1^{-m}). \tag{4.2.2}$$

Let us consider the Taylor expansion of $t \mapsto \phi(1, t)$ at t_0:

$$g(t) = \phi(1, t) = b_0 + (t - t_0) b_1 + \frac{1}{k!} (t - t_0)^k g_k(t),$$

where g_k is a smooth function such that $b_k = g_k(0)$. Thus, we get

$$\begin{aligned} \phi(x) &= x_1^{\mathcal{D}/\kappa_1} \left(b_0 + (x_2 \, x_1^{-m} - t_0) b_1 + \frac{1}{k!} (x_2 \, x_1^{-m} - t_0)^k g_k(x_2 \, x_1^{-m}) \right) \\ &= x_1^{\mathcal{D}/\kappa_1} (b_0 - t_0 b_1) + x_2 \, x_1^{(\mathcal{D} - \kappa_2)/\kappa_1} b_1 \\ &\quad + \frac{1}{k!} x_1^{(\mathcal{D} - k\kappa_2)/\kappa_1} (x_2 - t_0 x_1^m)^k g_k(x_2 \, x_1^{-m}). \end{aligned} \tag{4.2.3}$$

More generally, we have the formal series expansion:

$$\begin{aligned} \phi(x) &\approx \sum_{j=0}^{\infty} \frac{b_j}{j!} (x_2 - t_0 x_1^m)^j \, x_1^{\mathcal{D}/\kappa_1 - jm} \\ &= b_0 x_1^{\mathcal{D}/\kappa_1} + b_1 (x_2 - t_0 x_1^m) x_1^{\mathcal{D}/\kappa_1 - m} + \sum_{j=k}^{\infty} \frac{b_j}{j!} (x_2 - t_0 x_1^m)^j \, x_1^{\mathcal{D}/\kappa_1 - jm}. \end{aligned} \tag{4.2.4}$$

If $m = 1$ (i.e., $\kappa_1 = \kappa_2$) it will be usually better to write

$$\begin{aligned} \phi(x) &= x_1^{\mathcal{D}/\kappa_1} b_0 + (x_2 - t_0 x_1) x_1^{\mathcal{D}/\kappa_1 - 1} b_1 \\ &\quad + \frac{1}{k!} (x_2 - t_0 x_1)^k \, x_1^{\mathcal{D}/\kappa_1 - k} \, g_k(x_2 \, x_1^{-1}). \end{aligned} \tag{4.2.5}$$

Since $v_1 \sim 1$, we may assume

$$|x_1^{1/\kappa_1} - v_1^{1/\kappa_1}| \ll 1, \qquad |x_2 \, x_1^{-m} - v_2 \, v_1^{-m}| \ll 1.$$

The second condition is equivalent to $|x_2 - t_0 x_1^m| \ll 1$. Note that the points on the homogeneity curve through v satisfy the equation $x_2 = t_0 x_1^m$.

In order to determine the normal forms it will suffice to introduce three additional coordinate systems, which we shall denote by y, z, and w respectively, each having the point v as their origin. The original coordinate system is denoted by x. The function ϕ in the coordinate system y (resp. z, w) shall be denoted by ϕ^y (resp. ϕ^z, ϕ^w). For the original coordinate system x we simply use ϕ, or ϕ^x for emphasis.

The function ϕ in the coordinate system y (resp. z, w) but without the affine terms at v shall be denoted by ϕ_v^y (resp. ϕ_v^z, ϕ_v^w). This means

$$\phi_v^y(y) := \phi^y(y) - \phi^y(0) - y \cdot \nabla\phi^y(0),$$

and similarly for ϕ_v^z and ϕ_v^w.

The coordinate system y. It is defined through the following affine coordinate change having $v = (v_1, v_2)$ as the origin:

$$\begin{aligned} y_1 &= x_1 - v_1, \\ y_2 &= x_2 - v_2 - m v_2 v_1^{-1}(x_1 - v_1) \\ &= x_2 - (1-m)v_2 - m v_2 v_1^{-1} x_1. \end{aligned}$$

The reverse transformation is

$$\begin{aligned} x_1 &= y_1 + v_1, \\ x_2 &= y_2 + v_2 + m v_2 v_1^{-1} y_1. \end{aligned} \tag{4.2.6}$$

One can easily check that in these coordinates we can write

$$\begin{aligned} x_2 - t_0 x_1^m &= y_2 + v_2 + m v_2 v_1^{-1} y_1 - v_2(1 + v_1^{-1} y_1)^m \\ &= y_2 + v_2 + m v_2 v_1^{-1} y_1 - v_2\left(1 + m v_1^{-1} y_1 + \binom{m}{2} v_1^{-2} y_1^2 + \mathcal{O}(y_1^3)\right) \\ &= y_2 - y_1^2 \omega(y_1), \end{aligned}$$

i.e., the points on the homogeneity curve through v satisfy the equation $y_2 = y_1^2\omega(y_1)$ in y coordinates. Above (and in the following) we use the notation $\binom{c}{j} = c(c-1)\cdot\ldots\cdot(c-j+1)/j!$ for $c \in \mathbb{R}$ and j nonnegative integer. Furthermore, we obviously have:

Remark 4.2.1. *It holds that $\omega(0) \neq 0$ if and only if ω is not identically 0 if and only if $v_2 \neq 0$ (i.e., $t_0 \neq 0$) and $m \neq 1$.*

The coordinate system y will be used in most of the normal forms below which shall follow directly from the expression

$$\begin{aligned} \phi^y(y) &= (v_1 + y_1)^{\mathcal{D}/\kappa_1}(b_0 - t_0 b_1) \\ &\quad + (v_2 + y_2 + m v_2 v_1^{-1} y_1)\,(v_1 + y_1)^{(\mathcal{D}-\kappa_2)/\kappa_1} b_1 \\ &\quad + (y_2 - y_1^2\omega(y_1))^k r(y) \end{aligned} \tag{4.2.7}$$

which one obtains from (4.2.3) and (4.2.6). When $m = 1$ one uses (4.2.5) instead and gets

$$\phi^y(y) = (v_1 + y_1)^{\mathcal{D}/\kappa_1} b_0 + y_2(v_1 + y_1)^{\mathcal{D}/\kappa_1 - 1} b_1 + y_2^k r(y). \tag{4.2.8}$$

In both (4.2.7) and (4.2.8) the function r is smooth and nonvanishing at the origin. Let us also note that the expansion (4.2.4) can be rewritten in y coordinates as

$$\begin{aligned} \phi^y(y) &\approx b_0(v_1 + y_1)^{\mathcal{D}/\kappa_1} + b_1(y_2 - y_1^2\omega(y_1))(v_1 + y_1)^{\mathcal{D}/\kappa_1 - m} \\ &\quad + \sum_{j=k}^{\infty} \frac{b_j}{j!}(y_2 - y_1^2\omega(y_1))^j\,(v_1 + y_1)^{\mathcal{D}/\kappa_1 - jm}. \end{aligned} \tag{4.2.9}$$

The following simple lemma shall be useful later:

Lemma 4.2.2. *From equations (4.2.7) and (4.2.8) we get the following information on the second order derivatives of ϕ^y:*

(1) It always holds:

$$k = 2 \iff b_2 \neq 0 \iff \partial_2^2 \phi^y(0) \neq 0.$$

(2.a) If $\mathcal{D} \neq 1$ or $\kappa_2 \neq 1$ (i.e., $\mathcal{D} - \kappa_2 \neq 0$), then

$$b_1 \neq 0 \iff \partial_1 \partial_2 \phi^y(0) \neq 0.$$

(2.b) If $\mathcal{D} = \kappa_2 = 1$ or if $b_1 = 0$, then $\partial_1 \partial_2 \phi^y(0) = 0$.

(3.a) If $\mathcal{D} = 0$ and $\kappa_1 \neq \kappa_2$ (i.e., $m \neq 1$), or if $\mathcal{D} = \kappa_1 = 1$ and $\kappa_2 \neq 1$ (and in particular $m \neq 1$), then

$$b_1 \neq 0, t_0 \neq 0 \iff \partial_1^2 \phi^y(0) \neq 0,$$

and we remind that $v_2 \neq 0$ if and only if $t_0 \neq 0$.

(3.b) If $\mathcal{D} = \kappa_2 = 1$ and $\kappa_1 \neq 1$ (and in particular $m \neq 1$), then

$$b_0 - t_0 b_1 \neq 0 \iff \partial_1^2 \phi^y(0) \neq 0.$$

(3.c) If $m = 1$ (i.e., $\kappa_1 = \kappa_2$) or if $b_1 = 0$, then

$$b_0 \neq 0, \frac{\mathcal{D}}{\kappa_1} \notin \{0, 1\} \iff \partial_1^2 \phi^y(0) \neq 0.$$

Note that $\mathcal{D}/\kappa_1 = 0$ if and only if $\mathcal{D} = 0$, and $\mathcal{D}/\kappa_1 = 1$ if and only if $\mathcal{D} = \kappa_1 = 1$.

Proof. The only not completely trivial case is (3.a). Since in this case $\mathcal{D}/\kappa_1 \in \{0, 1\}$, the first term in (4.2.7) is an affine term, and so we can ignore it. Since $k \geqslant 2$, the third term also does not contribute to the y_1^2 term in the Taylor series of ϕ^y, and and so we can ignore it too. We therefore only need to consider the term:

$$(v_2 + y_2 + m v_2 v_1^{-1} y_1)\,(v_1 + y_1)^{(\mathcal{D}-\kappa_2)/\kappa_1} b_1,$$

and in fact, we may even reduce ourselves to

$$(v_2 + m v_2 v_1^{-1} y_1)\,(v_1 + y_1)^{(\mathcal{D}-\kappa_2)/\kappa_1} b_1 = b_1 v_2 (1 + m v_1^{-1} y_1)\,(v_1 + y_1)^{(\mathcal{D}-\kappa_2)/\kappa_1}.$$

Now if $t_0 = 0$ (i.e. $v_2 = 0$) or if $b_1 = 0$, then $\partial_1^2 \phi^y(0) = 0$ follows. Let us now assume $v_2 \neq 0$ and $b_1 \neq 0$. We note that in our case we may rewrite $(\mathcal{D} - \kappa_2)/\kappa_1 = \mathcal{D} - m$, and so it suffices to show that

$$\partial_{y_1}^2 \,|_{y_1=0} \left((1 + m v_1^{-1} y_1)\,(1 + v_1^{-1} y_1)^{\mathcal{D}-m} \right) \neq 0.$$

Calculating the second derivative one gets

$$2 m v_1^{-2} (\mathcal{D} - m) + v_1^{-2} (\mathcal{D} - m)(\mathcal{D} - m - 1).$$

This is not zero since in this case we have $\mathcal{D} \in \{0, 1\}$ and $m \notin \{0, 1\}$. $\square$

The coordinate systems z and w. These are defined through affine coordinate changes of the form

$$\begin{aligned} x_1 &= v_1 + z_1, & w_1 &= z_1 + \frac{1}{B} z_2, \\ x_2 &= v_2 + z_2 + A z_1, & w_2 &= z_2, \end{aligned} \tag{4.2.10}$$

having (v_1, v_2) as their origin, where we shall have $B := A - m v_2 v_1^{-1} \neq 0$ so that the coordinate system y never coincides with the coordinate system z, and the coordinate system z never coincides with the coordinate system w. The constant A shall depend on v and the first few derivatives of ϕ at v (note that $A = B \neq 0$ if $v_2 = t_0 = 0$). These coordinate systems will be described more precisely in Subsection 4.2.5. There we shall also introduce a smooth function $\tilde{\omega}$ such that

$$x_2 - t_0 x_1^m = y_2 - y_1^2 \omega(y_1) = (w_1 - w_2^2 \tilde{\omega}(w_2)) r_0(w)$$

for some smooth function r_0 satisfying $r_0(0) \neq 0$. Note that we have

$$\begin{aligned} y_1 &= z_1 = w_1 - \frac{1}{B} w_2, \\ y_2 &= z_2 + B z_1 = B w_1. \end{aligned} \tag{4.2.11}$$

Some general considerations regarding the Hessian determinant $\mathcal{H}_\phi$. Recall that

$$\phi(r^{\kappa_1} x_1, r^{\kappa_2} x_2) = r^{\mathcal{D}} \phi(x_1, x_2).$$

Taking derivatives in x_1 and x_2 we get

$$(\partial_1^{\tau_1} \partial_2^{\tau_2} \phi)(r^{\kappa_1} x_1, r^{\kappa_2} x_2) = r^{\mathcal{D} - \tau_1 \kappa_1 - \tau_2 \kappa_2} (\partial_1^{\tau_1} \partial_2^{\tau_2} \phi)(x_1, x_2).$$

Thus, we have for the Hessian determinant of ϕ:

$$\mathcal{H}_\phi(r^{\kappa_1} x_1, r^{\kappa_2} x_2) = r^{2(\mathcal{D} - |\kappa|)} \mathcal{H}_\phi(x_1, x_2).$$

From this it follows that if $\mathcal{H}_\phi$ vanishes at the point v, then it also vanishes along the homogeneity curve through v which we recall is parametrized by $r \mapsto (r^{\kappa_1} v_1, r^{\kappa_2} v_2)$.

We are interested in the order of vanishing of $\mathcal{H}_\phi$ in directions transversal to this curve. In particular, if we have $\partial_2^{\tau_2} \mathcal{H}_\phi(v) = 0$ for $\tau_2 < N$ and $\partial_2^N \mathcal{H}_\phi(v) \neq 0$, then by using homogeneity and a Taylor expansion (as we did for ϕ) we get

$$\mathcal{H}_\phi(x) = (x_2 - t_0 x_1^m)^N q(x),$$

for some smooth function q satisfying $q(v) \neq 0$. Calculating N shall be done in Subsection 4.2.6 by using the normal forms of ϕ. Recall that the Hessian determinant is equivariant under affine coordinate changes, and so we can freely change to y, z, or w coordinates.

Preliminary comments on the normal forms. Let us introduce the following notation for the nondegenerate case (i.e., the case when the Hessian determinant of ϕ does not vanish at v):

(ND) The function ϕ_v is nondegenerate at the origin.

When ϕ_v does not satisfy (ND), then we shall show that we can associate to it one of the following normal forms:

(i.y1) $\phi_v^y(y) = y_2^k r(y)$, $k \geqslant 2$,
$\mathcal{H}_{\phi^y}(y) = y_2^{\tilde{k}+2k-2} q(y)$, $0 \leqslant \tilde{k} \leqslant \infty$,

(i.y2) $\phi_v^y(y) = y_1^{\tilde{k}} r(y_1) + \varphi(y)$, $\tilde{k} \geqslant 2$,
φ and $\mathcal{H}_{\phi^y}$ are flat,

(i.w1) $\phi_v^w(w) = w_2^2 r(w_2) + \varphi(w)$,
φ and $\mathcal{H}_{\phi^w}$ are flat,

(i.w2) $\phi_v^w(w) = w_2^2 r(w) + \varphi(w)$,
$v_1 B \partial_1^j r(0) = jA(m-1)\partial_1^{j-1} r(0)$ for all $j \geqslant 1$,
where $A, B, v_1 \neq 0$ are defined as above,
φ and $\mathcal{H}_{\phi^w}$ are flat,

(ii.y) $\phi_v^y(y) = y_1^2 r_1(y_1) + y_2^k r_2(y)$, $k \geqslant 3$,
$\mathcal{H}_{\phi^y}(y) = y_2^{k-2} q(y)$,

(ii.w) $\phi_v^w(w) = w_1^{\tilde{k}} r_1(w) + w_2^2 r_2(w_2)$, $\tilde{k} \geqslant 3$,
$\mathcal{H}_{\phi^w}(w) = w_1^{\tilde{k}-2} q(w)$,

(iii) $\phi_v^w(w) = w_1^{\tilde{k}} r_1(w) + w_2^2 r_2(w)$, $\tilde{k} \geqslant 3$,
$v_1 B \partial_1^j r_2(0) = jA(m-1)\partial_1^{j-1} r_2(0)$ for $1 \leqslant j \leqslant \tilde{k}-1$,
where $A, B, v_1 \neq 0$ are defined as above,
$\mathcal{H}_{\phi^w}(w) = w_1^{\tilde{k}-2} q(w)$,

(iv) $\phi_v^y(y) = y_1^2 r_1(y_1) + (y_2 - y_1^2 \omega(y_1))^k r_2(y)$, $k \geqslant 3$,
$\mathcal{H}_{\phi^y}(y) = (y_2 - y_1^2 \omega(y_1))^{k-2} q(y)$,

(v) $\phi_v^w(w) = (w_1 - w_2^2 \tilde{\omega}(w_2))^{\tilde{k}} r_1(w) + w_2^2 r_2(w)$, $\tilde{k} \geqslant 3$,
$v_1 B \partial_1^j r_2(0) = jA(m-1)\partial_1^{j-1} r_2(0)$ for $1 \leqslant j \leqslant \tilde{k}-1$,
where $A, B, v_1 \neq 0$ are defined as above,
$\mathcal{H}_{\phi^w}(w) = (w_1 - w_2^2 \tilde{\omega}(w_2))^{\tilde{k}-2} q(w)$,

(vi) $\phi_v^y(y) = (y_2 - y_1^2 \omega(y_1))^k r(y)$, $k \geqslant 2$,
$\mathcal{H}_{\phi^y}(y) = (y_2 - y_1^2 \omega(y_1))^{2k-3} q(y)$.

All the appearing functions are smooth and do not vanish at the origin (except φ which is always flat). The number k is as defined in (4.2.1) and it is always finite in the above normal forms (when it is infinite it turns out that one is necessarily in case of Normal form (i.y2)). On the other hand, the definition of the number $\tilde{k}$ changes from case to case, and we allow $\tilde{k}$ to be infinite only in Normal form (i.y1), in which case we consider the Hessian determinant to be flat at the origin. Let us furthermore remark that Normal forms (i.w1) and (i.w2) stem from Normal forms (ii.w), (iii), and (v), in the sense that they correspond to $\tilde{k} = \infty$.

The first step in deriving the above normal forms is to switch to y coordinates. In most cases this will suffice and the normal form will be obvious, and so in the following subsections we shall leave out most of the details for these cases. In particular, as a consequence of considerations in Subsections 4.2.3 and 4.2.4, we shall obtain:

Lemma 4.2.3. *If $k \geqslant 3$ and if we are not in the (ND) case, then the function ϕ^y_v is always in one of the following normal forms: (i.y1), (i.y2), (ii.y), (iv), or (vi).*

If $k = 2$, $b_1 \neq 0$, $\mathcal{D} \neq \kappa_2$, and we are not in the (ND) case, then we shall either need to

(FP) Flip coordinates (i.e., exchange x_1 and x_2) and use the y coordinates associated to the flipped coordinates,

or we shall need w (and the intermediary z) coordinates. Details are to be found in Subsection 4.2.5 below.

Note that flipping coordinates makes sense only when $v_2 \neq 0$ (and indeed, we shall flip coordinates only when $A = 0$, which, as it turns out, never happens when $v_2 = 0$). After flipping coordinates it will always suffice to use the y coordinates (associated to the flipped x, v, and κ), and in particular, we shall be able to apply Lemma 4.2.3. Note that these y coordinates are not in general equal to flipped y coordinates associated to the original x, v, and κ.

4.2.2 Normal form when $t \mapsto \partial_2^2\phi(1,t)$ is flat at t_0 (i.e., $k = \infty$)

Let us assume that

$$\partial_2^j\phi(1, t_0) = 0 \quad \text{for all } j \geqslant 2, \tag{4.2.12}$$

and so we have $\partial_2^j\phi(v) = 0$ for all v (with $v_1 > 0$) satisfying $v_2 v_1^{-m} = t_0$ by (4.2.2). The Euler equation for ϕ is

$$\mathcal{D}\phi(x) = \kappa_1 x_1 \partial_1\phi(x) + \kappa_2 x_2 \partial_2\phi(x).$$

Taking the derivative $\partial^\tau = \partial_1^{\tau_1}\partial_2^{\tau_2}$ we get at (v_1, v_2) that

$$(\mathcal{D} - \kappa_1\tau_1 - \kappa_2\tau_2)\partial^\tau\phi(v) = \kappa_1 v_1 \partial^{\tau+(1,0)}\phi(v) + \kappa_2 v_2 \partial^{\tau+(0,1)}\phi(v).$$

From this, the fact that $\kappa_1 v_1 \neq 0$, and the flatness assumption (4.2.12) it follows by induction in τ_1 that $\partial^\tau\phi(v) = 0$ for all $\tau_1 \geqslant 0$ and $\tau_2 \geqslant 2$.

If now $\partial_1\partial_2\phi(v) \neq 0$, then the Hessian determinant does not vanish and we are in the (ND) case (this always happens for example when $\phi(x_1, x_2) = x_1 x_2$). On the other hand, if $\partial_1\partial_2\phi(v) = 0$, then we get in the same way as above that $\partial^\tau\phi(v) = 0$ for all $\tau_1 \geqslant 1$ and $\tau_2 = 1$. Thus, by using a Taylor expansion at v and by switching to y coordinates (recall $x_1 = y_1 + v_1$) we may write

$$\phi^y_v(y) = y_1^2 r(y_1) + \varphi(y),$$

where r is a smooth function and φ is a flat smooth function. In particular, in this case the Hessian determinant vanishes of infinite order at $x = v$ and therefore the condition (H2) cannot hold. This also shows that (H2) is a stronger condition than (H1). Since we assume that at least (H1) holds, then we necessarily have that $t \mapsto \partial_1^2 \phi(t, v_2)$ is not flat at v_1, and so r cannot be flat either, i.e., we can write

$$\phi_v^y(y) = y_1^{\tilde{k}} \tilde{r}(y_1) + \varphi(y),$$

for some smooth function $\tilde{r}$ satisfying $\tilde{r}(0) \neq 0$ and $\tilde{k} \geqslant 2$. This is precisely the Normal form (i.y2).

4.2.3 Normal form tables when ϕ mixed homogeneous of degree $\mathcal{D} = 0$

Recall that we assume $k < \infty$ in this and the following subsections. In this case (4.2.7) becomes

$$\phi^y(y) - (b_0 - t_0 b_1) = (v_2 + y_2 + m v_2 v_1^{-1} y_1)\,(v_1 + y_1)^{-m} b_1 + (y_2 - y_1^2 \omega(y_1))^k r(y)$$

if $m \neq 1$, and in the case $m = 1$ we have by (4.2.8) that

$$\phi^y(y) - b_0 = y_2 (v_1 + y_1)^{-1} b_1 + y_2^k r(y). \tag{4.2.13}$$

We have put the constant terms on the left hand side since we may freely ignore them. Note that in the case $m = 1$ we have $\partial_1^2 \phi^y(0) = 0$.

Case: $m = 1$**.**

Conditions	Case
$b_1 = 0$	Normal form (i.y1)
$b_1 \neq 0$	(ND)

Here we actually have in the case when $b_1 = 0$ a precise order of vanishing of the Hessian determinant: it is always $2k - 2$. This follows from Subsection 4.2.6 (see in particular (4.2.33)).

If $b_1 \neq 0$, then from (4.2.13) we obviously have $\partial_1 \partial_2 \phi^y(0) \neq 0$, and it follows that the Hessian determinant at 0 is nonzero.

Case: $m \neq 1$**.**

Conditions	Case
$t_0 = 0, b_1 = 0$	Normal form (i.y1)
$t_0 = 0, b_1 \neq 0$	(ND)
$t_0 \neq 0, b_1 = 0$	Normal form (vi)
$t_0 \neq 0, b_1 \neq 0, k \geqslant 3$	(ND)
$t_0 \neq 0, b_1 \neq 0, k = 2$	(ND), or (FP), or Normal form (v), or Normal form (i.w2)

In the case $t_0 = 0, b_1 \neq 0$ we apply Lemma 4.2.2, (2.a) and (3.a), and get respectively that $\partial_1\partial_2\phi^y(0) \neq 0$ and $\partial_1^2\phi^y(0) = 0$, from which it indeed follows that we are in the (ND) case. Similarly, in the case $t_0 \neq 0, b_1 \neq 0, k \geqslant 3$ we use Lemma 4.2.2, (1) and (2.a), and obtain that $\partial_2^2\phi^y(0) = 0$ and $\partial_1\partial_2\phi^y(0) \neq 0$, from which we again get that the Hessian determinant of ϕ^y does not vanish.

As the case $t_0 \neq 0, b_1 \neq 0, k = 2$ shall be treated in the same way as certain other cases which appear later and where w coordinates may be needed, we have postponed its discussion to Subsection 4.2.5.

4.2.4 Normal form tables when ϕ mixed homogeneous of degree $\mathcal{D} = \pm 1$

Recall that here we have

$$\phi(x) = x_1^{\mathcal{D}/\kappa_1}(b_0 - t_0 b_1) + x_2\, x_1^{(\mathcal{D}-\kappa_2)/\kappa_1} b_1 + \frac{1}{k!} x_1^{(\mathcal{D}-k\kappa_2)/\kappa_1}(x_2 - t_0 x_1^m)^k g_k(x_2\, x_1^{-m}) \tag{4.2.14}$$

and that in y coordinates this becomes

$$\begin{aligned}\phi^y(y) = {} & (v_1 + y_1)^{\mathcal{D}/\kappa_1}(b_0 - t_0 b_1) \\ & + (v_2 + y_2 + m v_2 v_1^{-1} y_1)\,(v_1 + y_1)^{(\mathcal{D}-\kappa_2)/\kappa_1} b_1 \\ & + (y_2 - y_1^2\omega(y_1))^k r(y).\end{aligned} \tag{4.2.15}$$

In this subsection (where $\mathcal{D} = \pm 1$) we need to consider five possible subcases. The cases we first consider are when $\mathcal{D} = \kappa_1$, or $\mathcal{D} = \kappa_2$, or both. Since κ_1 and κ_2 are strictly positive, these cases are only possible for $\mathcal{D} = 1$. The penultimate case is when $\kappa_1 = \kappa_2 \neq \mathcal{D}$, and the last case is when all of κ_1, κ_2, and $\mathcal{D}$ are different from each other.

Case: $\mathcal{D} = 1$, $\kappa_1 = 1$, $\kappa_2 = 1$.

In this case the first two terms in (4.2.15) become affine, and by Remark 4.2.1 we have $\omega \equiv 0$. As a consequence we have only one case:

Conditions	Case
-	Normal form (i.y1)

Furthermore, we note that initially we know that the order of vanishing of the Hessian determinant is at least $2k - 2$, which is always greater than or equal to 2. Since this is true at every point, the Hessian determinant vanishes identically in this case.

Case: $\mathcal{D} = 1$, $\kappa_1 \neq 1$, $\kappa_2 = 1$.

Here we first note that by Lemma 4.2.2, (2.b), we always have $\partial_1\partial_2\phi^y(0) = 0$. This is a simple consequence of the fact that in this case the second term in (4.2.15) is linear.

Conditions	Case
$b_0 - t_0 b_1 = 0, t_0 = 0$	Normal form (i.y1)
$b_0 - t_0 b_1 = 0, t_0 \neq 0$	Normal form (vi)
$b_0 - t_0 b_1 \neq 0, k = 2$	(ND)
$b_0 - t_0 b_1 \neq 0, k \geqslant 3, t_0 = 0$	Normal form (ii.y)
$b_0 - t_0 b_1 \neq 0, k \geqslant 3, t_0 \neq 0$	Normal form (iv)

The (ND) case follows from Lemma 4.2.2, (1) and (3.b).

Case: $\mathcal{D} = 1$, $\kappa_1 = 1$, $\kappa_2 \neq 1$.

Here we note that the first term in (4.2.15) becomes linear, and therefore does not influence the normal form of ϕ_v^y.

Conditions	Case
$t_0 = 0, b_1 = 0$	Normal form (i.y1)
$t_0 = 0, b_1 \neq 0$	(ND)
$t_0 \neq 0, b_1 = 0$	Normal form (vi)
$t_0 \neq 0, b_1 \neq 0, k \geqslant 3$	(ND)
$t_0 \neq 0, b_1 \neq 0, k = 2$	(ND) or (FP)

The cases $t_0 = 0, b_1 \neq 0$ and $t_0 \neq 0, b_1 \neq 0, k \geqslant 3$ are (ND) by the same argumentation as in the table above for $\mathcal{D} = 0$, $m \neq 1$ (namely, by applying Lemma 4.2.2, (2.a) and (3.a), in the case $t_0 = 0, b_1 \neq 0$, and by applying Lemma 4.2.2, (1) and (2.a), in the case $t_0 \neq 0, b_1 \neq 0, k \geqslant 3$).

Let us note the following for the last case where $t_0 \neq 0$, $b_1 \neq 0$, and $k = 2$. The expression in (4.2.14) can be rewritten as (after ignoring the first term, which is linear in this case):

$$b_1 x_2 \, x_1^{1-m} + \frac{b_2}{2} x_1^{1-2m} (x_2 - t_0 x_1^m)^2 + \mathcal{O}\Big((x_2 - t_0 x_1^m)^3\Big)$$

We want to calculate what the Hessian determinant of $\phi_v^x = \phi_v$ at v is (or equivalently, the Hessian determinant of ϕ at v). For this we only need the second derivatives of ϕ at v, and so we can freely ignore the last term of size $(x_2 - t_0 x_1^m)^3$. After expanding the second term in the above expression and ignoring the linear terms and the term $\mathcal{O}((x_2 - t_0 x_1^m)^3)$ we get

$$(b_1 - t_0 b_2) x_1^{1-m} x_2 + \frac{b_2}{2} x_1^{1-2m} x_2^2.$$

From this it follows by a direct calculation that

$$\partial_1^2 \phi(v) = -m \frac{v_2}{v_1} \partial_1 \partial_2 \phi(v),$$

and so

$$\mathcal{H}_\phi(v) = -\partial_1 \partial_2 \phi(v) \Big(\partial_1 \partial_2 \phi(v) + m \frac{v_2}{v_1} \partial_2^2 \phi(v) \Big),$$

which we note can be rewritten as

$$\mathcal{H}_\phi(v) = -\partial_1 \partial_2 \phi^x(v) \, \partial_1 \partial_2 \phi^y(0),$$

by (4.2.6). This implies in particular that $\mathcal{H}_\phi(v) = 0$ if and only if $\partial_1 \partial_2 \phi(v) = 0$ if and only if $\partial_1^2 \phi(v) = 0$ since by Lemma 4.2.2, (2.a), we know that $\partial_1 \partial_2 \phi^y(0) \neq 0$.

Thus, in the last case where $t_0 \neq 0$, $b_1 \neq 0$, and $k = 2$, we are either in the (ND) case, and otherwise we have $\partial_1^2 \phi(v) = 0$. This means precisely that the "k" associated to the flipped coordinates (and we can flip coordinates since $t_0 \neq 0$, i.e., $v_2 \neq 0$) is necessarily $\geqslant 3$. For the flipped coordinates we may now use the previous table where we have $\mathcal{D} = 1$, $\kappa_1 \neq 1$, $\kappa_2 = 1$ (or apply Lemma 4.2.3).

Case: $\mathcal{D} = \pm 1$, $\kappa_1 = \kappa_2 \neq \mathcal{D}$.

Here one uses (4.2.8):

$$\phi^y(y) = (v_1 + y_1)^{\mathcal{D}/\kappa_1} b_0 + y_2 (v_1 + y_1)^{\mathcal{D}/\kappa_1 - 1} b_1 + y_2^k r(y).$$

Conditions	Case
$b_0 = 0, b_1 = 0$	Normal form (i.y1)
$b_0 = 0, b_1 \neq 0$	(ND)
$b_0 \neq 0, b_1 = 0, k \geqslant 3$	Normal form (ii.y)
$b_0 \neq 0, b_1 = 0, k = 2$	(ND)
$b_0 \neq 0, b_1 \neq 0, k \geqslant 3$	(ND)
$b_0 \neq 0, b_1 \neq 0, k = 2$	(ND), or (FP), or Normal form (ii.w), or Normal form (i.w1)

The first (ND) case $b_0 = 0, b_1 \neq 0$ follows from Lemma 4.2.2, (2.a) and (3.c), the second (ND) case $b_0 \neq 0, b_1 = 0, k = 2$ follows from Lemma 4.2.2, (2.a), (3.c), and (1), and the third (ND) case $b_0 \neq 0, b_1 \neq 0, k \geqslant 3$ follows from Lemma 4.2.2, (1) and (2.a). For the last case $b_0 \neq 0, b_1 \neq 0, k = 2$ we again refer the reader to Subsection 4.2.5.

We give two further remarks. Firstly, one can show that in the case $b_0 = 0$, $b_1 = 0$ the order of vanishing of the Hessian determinant is precisely equal to $2k - 2$ if and only if we additionally have

$$\frac{\mathcal{D}}{\kappa_1} \notin \{1, k\},$$

as is shown in Subsection 4.2.6. Note that here we cannot have $\mathcal{D}/\kappa_1 = 1$, and when $\mathcal{D}/\kappa_1 = k$ from Subsection 4.2.6 we see that the Hessian determinant vanishes of order $2k + \tilde{k} - 2$ where $\tilde{k}$ is the smallest positive integer such that $b_{k+\tilde{k}} \neq 0$ (it is also possible $\tilde{k} = \infty$ with the obvious interpretation).

Secondly, here we can calculate explicitly from the derivatives $b_{\tau_2} = g^{(\tau_2)}(t_0)$ the number $\tilde{k}$ in the Normal form (ii.w) (see (4.2.28) in Subsection 4.2.5). This is already known for homogeneous polynomials [36].

Case: $\mathcal{D} = \pm 1$, $\kappa_1 \neq \mathcal{D}$, $\kappa_2 \neq \mathcal{D}$, $\kappa_1 \neq \kappa_2$.

Conditions	Case
$b_1 = 0, b_0 = 0, t_0 = 0$	Normal form (i.y1)
$b_1 = 0, b_0 = 0, t_0 \neq 0$	Normal form (vi)
$b_1 = 0, b_0 \neq 0, k = 2$	(ND)
$b_1 = 0, b_0 \neq 0, k \geqslant 3, t_0 = 0$	Normal form (ii.y)
$b_1 = 0, b_0 \neq 0, k \geqslant 3, t_0 \neq 0$	Normal form (iv)
$b_1 \neq 0, k \geqslant 3$	(ND)
$b_1 \neq 0, k = 2, t_0 = 0$	(ND), or Normal form (iii), or Normal form (i.w2)
$b_1 \neq 0, k = 2, t_0 \neq 0$	(ND), or (FP), or Normal form (v), or Normal form (i.w2)

The first (ND) case $b_1 = 0, b_0 \neq 0, k = 2$ follows from Lemma 4.2.2, (1), (2.a), and (3.c), and the second (ND) case $b_1 \neq 0, k \geqslant 3$ from Lemma 4.2.2, (1) and (2.a). For the very last two cases (namely, $b_1 \neq 0, k = 2, t_0 = 0$ and $b_1 \neq 0, k = 2, t_0 \neq 0$) we refer the reader, as usual, to Subsection 4.2.5.

4.2.5 The case when $\mathcal{D} \neq \kappa_2$, $b_1 \neq 0$, $k = 2$

In this subsection we shall discuss the remaining cases where y coordinates did not suffice and all of which (as one easily sees from the tables in the previous two subsection) satisfy $\mathcal{D} \neq \kappa_2$, $b_1 \neq 0$, $k = 2$. Here it will turn out that we are either in the (ND) case, or (FP) case, or that we need to use the w coordinates. In this case the form of the function ϕ in y coordinates is according to (4.2.7) equal to

$$\begin{aligned}\phi^y(y) = (v_1 + y_1)^{\mathcal{D}/\kappa_1}(b_0 - t_0 b_1) + (v_2 + y_2 + m v_2 v_1^{-1} y_1)\,(v_1 + y_1)^{(\mathcal{D}-\kappa_2)/\kappa_1} b_1 \\ + (y_2 - y_1^2\omega(y_1))^2 r(y),\end{aligned}$$

where $r(0) \neq 0$, and, as noted in Remark 4.2.1, $\omega \equiv 0$ if and only if $m = 1$ or $t_0 = 0$, and otherwise $\omega(0) \neq 0$. By Lemma 4.2.2, (1) and (2.a), we have

$$\partial_2^2 \phi^y(0) \neq 0 \qquad \text{and} \qquad \partial_1 \partial_2 \phi^y(0) \neq 0,$$

i.e., the y_2^2 term and the $y_1 y_2$ term in Taylor expansion of ϕ^y do not vanish. Therefore, depending on what the coefficient of the y_1^2 term is, it can happen that the Hessian determinant vanishes or not.

Case (ND) and the definition of z coordinates. If the Hessian determinant does not vanish, we are in the nondegenerate case. Otherwise, if the Hessian determinant does vanish, then since $\partial_2^2 \phi(v) \neq 0$ (which is by definition equivalent to $k = 2$), there is a coordinate system of the form

$$\begin{aligned} x_1 &= v_1 + z_1, \\ x_2 &= v_2 + z_2 + A z_1, \end{aligned}$$

with A unique, such that $\phi^x(x) = \phi^z(z)$, and such that the z_1^2 and $z_1 z_2$ terms in Taylor expansion of ϕ^z at the origin vanish, i.e.,

$$\partial_1^2 \phi^z(0) = 0 \qquad \text{and} \qquad \partial_1 \partial_2 \phi^z(0) = 0.$$

In particular, the coordinate systems y and z cannot coincide since the term $y_1 y_2$ does not vanish. This implies $B := A - m v_2 v_1^{-1} \neq 0$ (compare (4.2.6) and (4.2.10)).

Case (FP) and the reduction to $A \neq 0$. Let us now prove that we may reduce ourselves to the case

$$A \neq 0.$$

If $t_0 = 0$ (i.e., $v_2 = 0$), then we always have $A = B \neq 0$. The second possibility is $t_0 \neq 0$, and if in this case we would have $A = 0$, then z and x coordinates would coincide (up to a translation) which implies $\partial_{x_1}^2 \phi^x(v) = \partial_{z_1}^2 \phi^z(0) = 0$. Thus, by flipping coordinates, we would have that the k associated to the flipped coordinates is $\geqslant 3$, and so we would be in the case where the y coordinates associated to the flipped coordinates would suffice, i.e., we could apply Lemma 4.2.3.

This is also the reason why in the case when $\mathcal{D} = 1$, $\kappa_1 = 1$, and $\kappa_2 \neq 1$, it always sufficed to flip coordinates. The calculation below the corresponding table in Subsection 4.2.4 shows that $\mathcal{H}_\phi(v) = 0$ implies $\partial_1^2\phi(v) = \partial_1\partial_2\phi(v) = 0$, which in turn implies that one always has $A = 0$.

The normal form in z coordinates. Now that we may assume $A \neq 0$, our first step is to write down the Euler equation for homogeneous functions in z coordinates. The Euler equation is

$$\mathcal{D}\phi(x) = \kappa_1 x_1 \partial_1\phi(x) + \kappa_2 x_2 \partial_2 \phi(x).$$

By the definition of z coordinates we have

$$\begin{aligned}\partial_{x_1} &= \partial_{z_1} - A\partial_{z_2},\\ \partial_{x_2} &= \partial_{z_2}.\end{aligned}$$

Thus, the Euler equation in z coordinates is

$$\begin{aligned}\mathcal{D}\phi^z(z) &= \kappa_1(v_1 + z_1)\partial_1\phi^z(z)\\ &\quad - \kappa_1 A(v_1+z_1)\partial_2\phi^z(z) + \kappa_2(v_2 + z_2 + Az_1)\partial_2\phi^z(z)\\ &= \kappa_1(v_1+z_1)\partial_1\phi^z(z)\\ &\quad + \Big(-\kappa_1 v_1 B + A(-\kappa_1+\kappa_2)z_1 + \kappa_2 z_2\Big)\partial_2\phi^z(z). \end{aligned} \tag{4.2.16}$$

We now claim that if $\partial_1^{\tau_1+1}\phi^z(0) = \partial_1^{\tau_1}\partial_2\phi^z(0) = 0$ for all $1 \leqslant \tau_1 < N$ for some $N \geqslant 2$, then $\partial_1^{N+1}\phi^z(0) = 0$ if and only if $\partial_1^N\partial_2\phi^z(0) = 0$. But this is almost obvious. Namely, we just take the derivative ∂_1^N at 0 in the above Euler equation and get

$$\begin{aligned}\mathcal{D}\partial_1^N\phi^z(0) &= \kappa_1 v_1 \partial_1^{N+1}\phi^z(0) + \kappa_1 N \partial_1^N \phi^z(0)\\ &\quad - \kappa_1 v_1 B \partial_1^N \partial_2 \phi^z(0) + AN(-\kappa_1+\kappa_2)\partial_1^{N-1}\partial_2\phi^z(0).\end{aligned}$$

Using the assumption on vanishing derivatives we get

$$\partial_1^{N+1}\phi^z(0) = B\partial_1^N\partial_2\phi^z(0). \tag{4.2.17}$$

As we noted above $B \neq 0$ and our claim follows.

Now recall that $\partial_1^2\phi^z(0) = 0$ and $\partial_1\partial_2\phi^z(0) = 0$. Thus, the previously proved claim implies in particular by an inductive argument in N that either there is a $\tilde{k} \in \mathbb{N}$ such that $3 \leqslant \tilde{k} < \infty$, satisfying

$$\begin{aligned}\tilde{k} &= \min\{j \geqslant 2 : \partial_1^j\phi^z(0) \neq 0\}\\ &= \min\{j \geqslant 2 : \partial_1^{j-1}\partial_2\phi^z(0) \neq 0\},\end{aligned}$$

and

$$\phi_v^z(z) = z_1^{\tilde{k}} r_1(z) + z_1^{\tilde{k}-1} z_2 r_2(z) + z_2^2 r_3(z), \tag{4.2.18}$$

where $r_i(0) \neq 0$, $i = 1, 2, 3$, or that

$$\phi_v^z(z) = z_1^N r_{N,1}(z) + z_1^{N-1} z_2 r_{N,2}(z) + z_2^2 r_3(z),$$

for any $N \in \mathbb{N}$, which we shall consider as the case when $\tilde{k} = \infty$.

The normal form in w coordinates. It will be advantageous to use w coordinates where unlike in (4.2.18) the $w_1^{\tilde{k}-1} w_2$ term is no longer present, i.e., that we may write:

$$\phi_v^w(w) = w_1^{\tilde{k}} r_1(w) + w_2^2 r_2(w). \tag{4.2.19}$$

This fact follows directly from (4.2.17) and from

$$\begin{aligned} \partial_{w_1} &= \partial_{z_1}, \\ \partial_{w_2} &= \partial_{z_2} - \frac{1}{B} \partial_{z_1}, \end{aligned}$$

which we get from the definition of w coordinates (4.2.10). Actually, we can gain more information, especially in the case when $m = 1$. To see this let us rewrite the Euler equation in w coordinates by using (4.2.16):

$$\begin{aligned} \frac{\mathcal{D}}{\kappa_1} \phi^w(w) &= \Big(v_1 + w_1 - \frac{1}{B} w_2 \Big) \partial_1 \phi^w(w) \\ &\quad + \Big(- v_1 B + A(m-1)(w_1 - \frac{1}{B} w_2) + m w_2 \Big) (\partial_2 + \frac{1}{B} \partial_1) \phi^w(w) \\ &= \Big(\frac{B + A(m-1)}{B} w_1 + \frac{(B-A)(m-1)}{B^2} w_2 \Big) \partial_1 \phi^w(w) \\ &\quad + \Big(- v_1 B + A(m-1) w_1 + \frac{Bm - A(m-1)}{B} w_2 \Big) \partial_2 \phi^w(w). \end{aligned}$$

Case $m = 1$**.** Here the Euler equation reduces to

$$\frac{\mathcal{D}}{\kappa_1} \phi^w(w) = w_1 \partial_1 \phi^w(w) + (-v_1 B + w_2) \partial_2 \phi^w(w). \tag{4.2.20}$$

Taking the $\partial^\tau = \partial_1^{\tau_1} \partial_2^{\tau_2}$ derivative and evaluating at 0 one gets

$$\frac{\mathcal{D}}{\kappa_1} \partial^\tau \phi^w(0) = \tau_1 \partial^\tau \phi^w(0) - v_1 B \partial_1^{\tau_1} \partial_2^{\tau_2+1} \phi^w(0) + \tau_2 \partial^\tau \phi^w(0),$$

which can be rewritten as

$$\Big(\frac{\mathcal{D}}{\kappa_1} - |\tau| \Big) \partial^\tau \phi^w(0) = -v_1 B \partial_1^{\tau_1} \partial_2^{\tau_2+1} \phi^w(0).$$

From this and the fact from (4.2.19) that $\partial^\tau \phi^w(0) = 0$ for all τ satisfying $|\tau| = \tau_1 + \tau_2 \geqslant 2$, $0 \leqslant \tau_1 \leqslant \tilde{k} - 1$, and $0 \leqslant \tau_2 \leqslant 1$, one easily gets by induction on τ_2 that

$$\partial_1^{\tau_1}\partial_2^{\tau_2}\phi^w(0) = 0 \quad \text{when} \quad |\tau| = \tau_1 + \tau_2 \geqslant 2,\ 1 \leqslant \tau_1 \leqslant \tilde{k} - 1. \tag{4.2.21}$$

We may actually prove a stronger claim, namely that

$$\begin{aligned} \partial_1^{\tau_1}\phi^w(0, w_2) &\equiv 0, \qquad\qquad \text{for } 2 \leqslant \tau_1 \leqslant \tilde{k} - 1, \\ \partial_1\phi^w(0, w_2) &\equiv \partial_1\phi^w(0). \end{aligned} \tag{4.2.22}$$

In order to obtain this we take the $\partial_1^{\tau_1}$ derivative in (4.2.20) and evaluate it at $(0, w_2)$ to get

$$\Big(\frac{\mathcal{D}}{\kappa_1} - \tau_1\Big)\partial_1^{\tau_1}\phi^w(0, w_2) = (-v_1 B + w_2)\partial_2\partial_1^{\tau_1}\phi^w(0, w_2).$$

We note that this is a simple ordinary differential equation in w_2 of first order. It has a unique solution for $2 \leqslant \tau_1 \leqslant \tilde{k}-1$ since $-v_1B+w_2 \neq 0$ for small w_2, and since we can take (4.2.21) as initial conditions. The claim for $2 \leqslant \tau_1 \leqslant \tilde{k} - 1$ follows since $\partial_1^{\tau_1}\phi^w(0, w_2) \equiv 0$ is obviously a solution. For $\tau_1 = 1$ we note that the case $\mathcal{D}/\kappa_1 - \tau_1 = 0$ is trivial, and the solution is a unique constant function (necessarily equal to $\partial_1\phi^w(0)$). When $\tau_1 = 1$ and $\mathcal{D}/\kappa_1 - \tau_1 \neq 0$, then the differential equation evaluated at $w_2 = 0$ gives us that $\partial_1\partial_2\phi^w(0) = 0$ implies $\partial_1\phi^w(0) = 0$, which again means that $\partial_1\phi^w(0, w_2) \equiv 0$ is the unique solution of the given differential equation. We have thus proven (4.2.22).

Now by using Taylor approximation in w_1 for a fixed w_2, and the just proven fact for the mapping $w_2 \mapsto \partial_1^{\tau_1}\phi^w(0, w_2)$ for $1 \leqslant \tau_1 \leqslant \tilde{k} - 1$, we obtain that the normal form of ϕ^w (4.2.19) in the case $m = 1$ can be rewritten as

$$\phi_v^w(w) = w_1^{\tilde{k}} r_1(w) + w_2^2 r_2(w_2),$$

where $r_1(0), r_2(0) \neq 0$. Note that now r_2 depends only on w_2. This corresponds to Normal form (ii.w) when $\tilde{k}$ is finite and to Normal form (i.w1) otherwise.

Case $m \neq 1$**.** In this case we use our assumption that $A \neq 0$ in a critical way. Here it will be important to know what happens with $\partial_1^{\tau_1}\partial_2^2\phi^w(0)$ for $0 \leqslant \tau_1 \leqslant \tilde{k} - 1$, and also how one can rewrite the normal form of the Hessian determinant $\mathcal{H}_{\phi^w}$ (and in particular its root).

Let us begin by taking the $\partial_1^{\tau_1}\partial_2$ derivative of the Euler equation in w coordinates and evaluating it at $w = 0$. One gets

$$\begin{aligned} \frac{\mathcal{D}}{\kappa_1}\partial_1^{\tau_1}\partial_2\phi^w(0) = {} & \tau_1\frac{B + A(m-1)}{B}\partial_1^{\tau_1}\partial_2\phi^w(0) + \frac{(B-A)(m-1)}{B^2}\partial_1^{\tau_1+1}\phi^w(0) \\ & - v_1 B\partial_1^{\tau_1}\partial_2^2\phi^w(0) + \tau_1 A(m-1)\partial_1^{\tau_1-1}\partial_2^2\phi^w(0) \\ & + \frac{Bm - A(m-1)}{B}\partial_1^{\tau_1}\partial_2\phi^w(0). \end{aligned}$$

Now recall again from (4.2.19) that $\partial^\tau\phi^w(0) = 0$ holds for any τ satisfying $|\tau| = \tau_1 + \tau_2 \geqslant 2$, $0 \leqslant \tau_1 \leqslant \tilde{k} - 1$, and $0 \leqslant \tau_2 \leqslant 1$. Thus, if $1 \leqslant \tau_1 \leqslant \tilde{k} - 2$ then we get

$$v_1 B\partial_1^{\tau_1}\partial_2^2\phi^w(0) = \tau_1 A(m-1)\partial_1^{\tau_1-1}\partial_2^2\phi^w(0), \tag{4.2.23}$$

and if $\tau_1 = \tilde{k} - 1$, then

$$v_1 B \partial_1^{\tilde{k}-1} \partial_2^2 \phi^w(0) = \frac{(B-A)(m-1)}{B^2} \partial_1^{\tilde{k}} \phi^w(0) + (\tilde{k}-1) A(m-1) \partial_1^{\tilde{k}-2} \partial_2^2 \phi^w(0),$$

i.e., since $B - A = -m v_2 v_1^{-1}$, we can rewrite this as

$$v_1 B \partial_1^{\tilde{k}-1} \partial_2^2 \phi^w(0) + \frac{v_2 m(m-1)}{v_1 B^2} \partial_1^{\tilde{k}} \phi^w(0) = (\tilde{k}-1) A(m-1) \partial_1^{\tilde{k}-2} \partial_2^2 \phi^w(0). \tag{4.2.24}$$

Now since $A, B, v_1 \neq 0$, and $m \neq 1$, from (4.2.23) we may conclude by induction on τ_1 that for $0 \leqslant \tau_1 \leqslant \tilde{k} - 2$ one has

$$\partial_1^{\tau_1} \partial_2^2 \phi^w(0) \neq 0.$$

In order to unravel what is happening with $\partial_1^{\tilde{k}-1} \partial_2^2 \phi^w(0)$ we need to investigate the root of $\mathcal{H}_{\phi^w}$. For this we want to solve the equation

$$\begin{aligned} x_2 - t_0 x_1^m &= y_2 - \binom{m}{2} v_1^{-2} v_2 y_1^2 + \mathcal{O}(y_1^3) \\ &= 0. \end{aligned}$$

in the w coordinates, representing the homogeneity curve through v. Recall that by (4.2.11) we have $y_1 = w_1 - w_2/B$, $y_2 = B w_1$, and so we want to solve

$$B w_1 - \binom{m}{2} v_1^{-2} v_2 (w_1 - \frac{1}{B} w_2)^2 + \mathcal{O}((w_1 - \frac{1}{B} w_2)^3) = 0$$

for the w_1 variable in terms of the w_2 variable when $|w_1|, |w_2|$ are small numbers. Using the above equation one gets by a simple calculation that

$$\begin{aligned} w_1 &= \frac{v_2 m(m-1)}{2 v_1^2 B^3} w_2^2 + \mathcal{O}(w_2^3) \\ &= w_2^2 \tilde{\omega}(w_2), \end{aligned} \tag{4.2.25}$$

and $\tilde{\omega} \equiv 0$ if and only if $v_2 = 0 = t_0$. Note that we have the precise value of $\tilde{\omega}(0)$. Using this we can now write down the normal form of w as

$$\begin{aligned} \phi_v^w(w) &= w_1^{\tilde{k}} r_1(w) + w_2^2 r_2(w) \\ &= (w_1 - w_2^2 \tilde{\omega}(w_2))^{\tilde{k}} r_1(w) \\ &\quad + w_2^2 \Big(r_2(w) + \tilde{k} w_1^{\tilde{k}-1} \frac{v_2 m(m-1)}{2 v_1^2 B^3} r_1(w) \Big) + \mathcal{O}(w_2^4) \\ &= (w_1 - w_2^2 \tilde{\omega}(w_2))^{\tilde{k}} \tilde{r}_1(w) + w_2^2 \tilde{r}_2(w), \end{aligned} \tag{4.2.26}$$

where one can easily check by using (4.2.23), (4.2.24), (4.2.25), and (4.2.26) that $\partial_1^{\tau_1} \tilde{r}_2(0) \neq 0$ for all $0 \leqslant \tau_1 \leqslant \tilde{k} - 1$, and that in fact one has the relations

$$v_1 B \partial_1^{\tau_1} \tilde{r}_2(0) = \tau_1 A(m-1) \partial_1^{\tau_1 - 1} \tilde{r}_2(0)$$

for $1 \leqslant \tau_1 \leqslant \tilde{k}-1$. If $\tilde{k}=\infty$, then the above normal form in (4.2.26) corresponds to Normal form (i.w2). Otherwise we have $3 \leqslant \tilde{k} < \infty$ and two subcases. Namely, if $t_0 \neq 0$ (i.e., $\tilde{\omega}(0) \neq 0$), then the above normal form corresponds to Normal form (v), and if $t_0 = 0$ (and therefore $\tilde{\omega} \equiv 0$), then it corresponds to Normal form (iii).

Determining $\tilde{k}$ in the special case when $\mathcal{D} = \pm 1$ and $\kappa_1 = \kappa_2 \neq \mathcal{D}$. According to the corresponding table for this case in Subsection 4.2.4 here we may assume $b_0, b_1 \neq 0$, and note that here $m = 1$. We prove that the Hessian determinant of ϕ vanishes at v if and only if

$$b_2 = (1-\mathcal{D}\kappa_1)\frac{b_1^2}{b_0} = \Big(1-\frac{\kappa_1}{\mathcal{D}}\Big)\frac{b_1^2}{b_0}. \tag{4.2.27}$$

In this case we furthermore have that if $\tilde{k} < \infty$, then

$$\begin{aligned} b_j &= (\mathcal{D}\kappa_1)^j j! \binom{\mathcal{D}/\kappa_1}{j} \frac{b_1^j}{b_0^{j-1}}, \qquad && \text{for } j = 2, \dots, \tilde{k}-1 \\ b_{\tilde{k}} &\neq (\mathcal{D}\kappa_1)^{\tilde{k}} \tilde{k}! \binom{\mathcal{D}/\kappa_1}{\tilde{k}} \frac{b_1^{\tilde{k}}}{b_0^{\tilde{k}-1}}, \end{aligned} \tag{4.2.28}$$

and if $\tilde{k} = \infty$, then

$$b_j = (\mathcal{D}\kappa_1)^j j! \binom{\mathcal{D}/\kappa_1}{j} \frac{b_1^j}{b_0^{j-1}}, \qquad \text{for } j \in \{2, 3, \dots\}.$$

These formulae have already been shown for homogeneous polynomials in [36, Lemma 2.2]. Therefore, we only sketch how one can prove them in our slightly more general case.

Recall from (4.2.9) that we have the formal series for ϕ at $y = 0$:

$$\begin{aligned} \phi^y(y) &\approx (v_1+y_1)^{\frac{\mathcal{D}}{\kappa_1}} b_0 + y_2 (v_1+y_1)^{\frac{\mathcal{D}}{\kappa_1}-1} b_1 + \frac{1}{2!}(v_1+y_1)^{\frac{\mathcal{D}}{\kappa_1}-2} y_2^2 b_2 + \dots \\ &= \sum_{j=0}^{\infty} \frac{b_j}{j!} (v_1+y_1)^{\frac{\mathcal{D}}{\kappa_1}-j} y_2^j. \end{aligned}$$

From this one gets

$$\partial_1^2 \phi^y(0) = b_0 \frac{\mathcal{D}}{\kappa_1}\Big(\frac{\mathcal{D}}{\kappa_1}-1\Big) v_1^{\frac{\mathcal{D}}{\kappa_1}-2}, \quad \partial_1\partial_2 \phi^y(0) = b_1 \Big(\frac{\mathcal{D}}{\kappa_1}-1\Big) v_1^{\frac{\mathcal{D}}{\kappa_1}-2}, \quad \partial_2^2 \phi^y(0) = b_2 v_1^{\frac{\mathcal{D}}{\kappa_1}-2},$$

and (4.2.27) follows by a direct computation (recall that $\mathcal{H}_{\phi^y}(0) = 0$ if and only if $\mathcal{H}_\phi(v) = 0$). More generally, we have

$$\partial^\tau \phi^y(0) = \tau_1! \binom{\frac{\mathcal{D}}{\kappa_1} - \tau_2}{\tau_1} v_1^{\frac{\mathcal{D}}{\kappa_1}-|\tau|} b_{\tau_2}. \tag{4.2.29}$$

Let us now determine the relation between y and z when the Hessian determinant vanishes. We may write

$$\begin{aligned} z_1 &= y_1, \qquad & \partial_{z_1} &= \partial_{y_1} + B\partial_{y_2}, \\ z_2 &= y_2 - By_1, & \partial_{z_2} &= \partial_{y_2}. \end{aligned}$$

Then by (4.2.27) one gets that $\partial_1^2\phi^z(0) = \partial_1\partial_2\phi^z(0) = 0$ if and only if

$$B = -\frac{b_0}{b_1}\frac{\mathcal{D}}{\kappa_1}.$$

From this we can determine the constant A since it is equal to $t_0 + B$, i.e., $A = v_2/v_1 - (\mathcal{D}b_0)/(\kappa_1 b_1)$.

One can now directly prove (4.2.28) by induction in j by using (4.2.29), and the fact that $\partial_1^j\phi^z(0) = 0$ for $2 \leqslant j < \tilde{k}$ and $\partial_1^{\tilde{k}}\phi^z(0) \neq 0$ is equivalent to

$$\begin{aligned} &\Big(\partial_1 - \frac{b_0}{b_1}\frac{\mathcal{D}}{\kappa_1}\partial_2\Big)^j \phi^y(0) = 0, \qquad j = 2, \dots, \tilde{k}-1, \\ &\Big(\partial_1 - \frac{b_0}{b_1}\frac{\mathcal{D}}{\kappa_1}\partial_2\Big)^{\tilde{k}} \phi^y(0) \neq 0. \end{aligned}$$

We have already checked the induction base $j = 2$.

4.2.6 Order of vanishing of the Hessian determinant

In this subsection we determine the normal forms of the Hessian determinant of ϕ (or more precisely, the order of vanishing of the Hessian determinant of ϕ), as listed in Subsection 4.2.1. We recall from Subsection 4.2.1 that if $v_1 > 0$, then one can write

$$\mathcal{H}_\phi(x) = (x_2 - t_0 x_1^m)^N q(x),$$

where either q is flat in v (which we consider as the case $N = \infty$), or $q(v) \neq 0$ and $0 \leqslant N < \infty$. It remains to determine N from the information provided by the normal forms of ϕ. We note that

$$N = \min\{j \geqslant 0 : (\partial_2^j\mathcal{H})(v) \neq 0\}.$$

Normal form (i.y1). First we note by the normal form tables above that this normal form appears only in cases when either $m = 1$ or $t_0 = v_2 = 0$, and so we have $\omega \equiv 0$. Thus, by (4.2.9) the function ϕ_v^y has the formal expansion:

$$\begin{aligned} \phi_v^y(y) &= \frac{1}{k!} y_2^k \,(y_1 + v_1)^{\mathcal{D}/\kappa_1 - km}\, g_k(y_2(y_1+v_1)^{-1} + t_0) \\ &\approx \sum_{j=k}^{\infty} \frac{b_j}{j!} y_2^j \,(y_1+v_1)^{\mathcal{D}/\kappa_1 - jm}, \end{aligned} \tag{4.2.30}$$

and the Hessian determinant vanishes along $y_2 = 0$, which means we need to determine what is the least N such that $(\partial_2^N\mathcal{H}_{\phi^y})(0) \neq 0$. From the above expansion one obtains

$$\begin{aligned} \partial_1^{\tau_1}\partial_2^{\tau_2}\phi^y(0) &= 0, \qquad |\tau| = \tau_1 + \tau_2 \geqslant 2,\ 0 \leqslant \tau_2 \leqslant k-1, \\ \partial^\tau\phi^y(0) &= \tau_1! \binom{\frac{\mathcal{D}}{\kappa_1} - m\tau_2}{\tau_1} v_1^{\frac{\mathcal{D}}{\kappa_1} - \tau_1 - m\tau_2} b_{\tau_2}, \qquad \tau_2 \geqslant k. \end{aligned} \tag{4.2.31}$$

By applying the general Leibniz rule to the definition of the Hessian determinant we get

$$\begin{aligned}\partial_2^N \mathcal{H}_{\phi^y} &= \partial_2^N (\partial_1^2\phi^y \partial_2^2\phi^y - (\partial_1\partial_2\phi^y)^2) \\ &= \sum_{n=0}^{N} \binom{N}{n} \Big(\partial_1^2\partial_2^n\phi^y \, \partial_2^{N+2-n}\phi^y - \partial_1\partial_2^{n+1}\phi^y \, \partial_1\partial_2^{N+1-n}\phi^y \Big),\end{aligned} \tag{4.2.32}$$

and one can easily check by using (4.2.31) that $\partial_2^N \mathcal{H}_{\phi^y}(0) = 0$ for $N < 2k-2$. For $N = 2k-2$ we get

$$\begin{aligned}&\partial_2^{2k-2}\mathcal{H}_{\phi^y}(0) \\ &= \binom{2k-2}{k} \partial_1^2\partial_2^k\phi^y(0)\partial_2^k\phi^y(0) - \binom{2k-2}{k-1}(\partial_1\partial_2^k\phi^y)^2(0) \\ &= \left[\binom{2k-2}{k}\Big(\frac{\mathcal{D}}{\kappa_1} - km\Big)\Big(\frac{\mathcal{D}}{\kappa_1} - km - 1\Big) - \binom{2k-2}{k-1}\Big(\frac{\mathcal{D}}{\kappa_1} - km\Big)^2\right] b_k^2 v_1^{2\frac{\mathcal{D}}{\kappa_1} - 2km - 2} \\ &= \left[\frac{k-1}{k}\Big(\frac{\mathcal{D}}{\kappa_1} - km - 1\Big) - \Big(\frac{\mathcal{D}}{\kappa_1} - km\Big)\right]\binom{2k-2}{k-1}\Big(\frac{\mathcal{D}}{\kappa_1} - km\Big) b_k^2 v_1^{2\frac{\mathcal{D}}{\kappa_1} - 2km - 2}.\end{aligned}$$

Thus, $\partial_2^{2k-2}\mathcal{H}_{\phi^y}(0) \neq 0$ if and only if

$$\frac{\mathcal{D}}{\kappa_1} \notin \Big\{ km,\ km + 1 - k \Big\}. \tag{4.2.33}$$

Let us now additionally assume that

$$\begin{aligned} b_{k+j} &= 0, \qquad 0 < j < \tilde{k}, \\ b_{k+\tilde{k}} &\neq 0, \end{aligned}$$

for some $\tilde{k} \geqslant 1$.

Case when $\frac{\mathcal{D}}{\kappa_1} = km$. By examining the term $j = k$ in (4.2.30) we note that in this case we additionally have

$$\begin{aligned} \partial_2^k\phi^y(0) &\neq 0 \\ \partial_1^{\tau_1}\partial_2^k\phi^y(0) &= 0, \qquad \tau_1 \geqslant 1. \end{aligned}$$

Now by using the information in (4.2.31), the above additional assumption that $b_{k+j} = 0$ for $0 < j < \tilde{k}$, $b_{k+\tilde{k}} \neq 0$, and the Leibniz formula (4.2.32) a straightforward calculation yields that $\partial_2^N\mathcal{H}_{\phi^y}(0) = 0$ for $N < 2k + \tilde{k} - 2$ and $\partial_2^{2k+\tilde{k}-2}\mathcal{H}_{\phi^y}(0) \neq 0$, i.e., we have the precise order of vanishing of the Hessian determinant.

Case when $\frac{\mathcal{D}}{\kappa_1} = km+1-k$. Again, by a straightforward calculation using the Leibniz formula one gets that $\partial_2^N\mathcal{H}_{\phi^y}(0) = 0$ for $N < 2k + \tilde{k} - 2$ and we have for $N = 2k + \tilde{k} - 2$:

$$\begin{aligned}\partial_2^{2k+\tilde{k}-2}\mathcal{H}_{\phi^y}(0) = &\binom{2k+\tilde{k}-2}{k}\partial_1^2\partial_2^k\phi^y(0)\partial_2^{k+\tilde{k}}\phi^y(0) \\ &+ \binom{2k+\tilde{k}-2}{k-2}\partial_1^2\partial_2^{k+\tilde{k}}\phi^y(0)\partial_2^k\phi^y(0) \\ &- 2\binom{2k+\tilde{k}-2}{k-1}\partial_1\partial_2^k\phi^y(0)\,\partial_1\partial_2^{k+\tilde{k}}\phi^y(0).\end{aligned}$$

Thus

$$\left(\binom{2k+\tilde{k}-2}{k-2}\right)^{-1}\partial_2^{2k+\tilde{k}-2}\mathcal{H}_{\phi^y}(0)=\frac{(k+\tilde{k})(k+\tilde{k}-1)}{(k-1)k}\partial_1^2\partial_2^k\phi^y(0)\partial_2^{k+\tilde{k}}\phi^y(0)$$
$$+\partial_1^2\partial_2^{k+\tilde{k}}\phi^y(0)\partial_2^k\phi^y(0)$$
$$-\frac{2(k+\tilde{k})}{k-1}\partial_1\partial_2^k\phi^y(0)\,\partial_1\partial_2^{k+\tilde{k}}\phi^y(0).$$

This is equal to zero when the expression

$$(k+\tilde{k})(k+\tilde{k}-1)\partial_1^2\partial_2^k\phi^y(0)\partial_2^{k+\tilde{k}}\phi^y(0)$$
$$+(k-1)k\partial_1^2\partial_2^{k+\tilde{k}}\phi^y(0)\partial_2^k\phi^y(0)-2k(k+\tilde{k})\partial_1\partial_2^k\phi^y(0)\,\partial_1\partial_2^{k+\tilde{k}}\phi^y(0)$$

equals zero. Plugging in the values of the derivatives from (4.2.31) one obtains that the above expression equals to

$$(k+\tilde{k})(k+\tilde{k}-1)(1-k)(-k)+(k-1)k(1-k-\tilde{k}m)(-k-\tilde{k}m)$$
$$-2k(k+\tilde{k})(1-k)(1-k-\tilde{k}m),$$

up to a nonzero constant factor. Factoring out $(1-k)(-k)$ we get

$$(k+\tilde{k})(k+\tilde{k}-1)+(k+\tilde{k}m-1)(k+\tilde{k}m)-2(k+\tilde{k})(k+\tilde{k}m-1)$$

and this equals zero if and only if $m\in\{1,(\tilde{k}+1)/\tilde{k}\}$.

The condition $\frac{\mathcal{D}}{\kappa_1}=km+1-k$ tells us that if $m=1$ then $\mathcal{D}=\kappa_1=\kappa_2=1$, and from the normal form tables we see that this is precisely when the Hessian determinant vanishes of infinite order.

In the case $m=(\tilde{k}+1)/\tilde{k}$ we get that $\mathcal{D}=1$, $\kappa_1=\tilde{k}/(k+\tilde{k})$ and $\kappa_2=(\tilde{k}+1)/(k+\tilde{k})$. It seems that in this case the order of vanishing of the Hessian determinant depends explicitly on the values b_j, and so, in contrast to the previous cases, one cannot relate in an easy way the order of vanishing of the Hessian determinant and the form of ϕ in (4.2.30). As we shall not need the precise order of vanishing of the Hessian determinant in this case, we do not pursue this question further.

Other normal forms. First we recall that Normal form (i.y2) was dealt with in Subsection 4.2.2, and there it was already determined that the Hessian vanishes of infinite order (i.e., it is flat).

In all the remaining normal forms we use either y or w coordinates, and so (as already noted in Subsection 4.2.1) the Hessian determinant in these coordinates has the normal form

$$\mathcal{H}_{\phi^u}(u)=(u_2-u_1^2\psi(u_1))^N q(u),$$

where u can represent either y or w coordinates, and where either N is finite and $q(0)\neq 0$, or the Hessian determinant is flat (in which case we consider N to be infinite). The function ψ is equal to either ω or $\tilde{\omega}$. Our goal is to determine $N=\min\{j\geqslant 0:(\partial_2^j\mathcal{H}_{\phi^u})(0)\neq 0\}$.

We first note that we can rewrite all the remaining normal forms as either

$$\phi_v^u(u) = (u_2 - u_1^2\psi(u_1))^{k_0} r(u) \tag{4.2.34}$$

or

$$\phi_v^u(u) = u_1^2 r_1(u) + u_2^{k_0} r_2(u), \tag{4.2.35}$$

where $r(0), \psi(0), r_1(0), r_2(0) \neq 0$, and $k_0 \geqslant 2$ in the first case and $k_0 \geqslant 3$ in the second. In the second case $k_0 = \infty$ is allowed with an obvious interpretation. Note that the second case (4.2.35) includes Normal forms (ii), (iii), (iv), (v), and also subcases of (i) where the w coordinates are used.

For both cases (4.2.34) and (4.2.35) one can use the Leibniz rule (4.2.32) and the information on the Taylor series of ϕ_v^u gained from these normal forms to obtain the order of vanishing of the Hessian determinant (in the ∂_{u_2} direction) by a direct calculation. In the first case (4.2.34) one gets that the order of vanishing is $N = 2k_0 - 3$ and in the second case (4.2.35) one gets that $N = k_0 - 2$ (or that the Hessian determinant is flat if $k_0 = \infty$).

4.3 Fourier restriction when a mitigating factor is present

In this section we prove Theorem 1.4.1, i.e., the Fourier restriction estimate

$$\|\widehat{f}\|_{L^2(\mathrm{d}\mu)} \leqslant C\|f\|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})},$$

where μ is the surface carried measure

$$\langle \mu, f\rangle = \int_{\mathbb{R}^2\setminus\{0\}} f(x, \phi(x))\, |\mathcal{H}_\phi(x)|^{\mathfrak{s}}\, \mathrm{d}x$$

and the exponents are

$$\Big(\frac{1}{\mathfrak{p}_1'}, \frac{1}{\mathfrak{p}_3'}\Big) = \Big(\frac{1}{2} - \mathfrak{s}, \mathfrak{s}\Big).$$

We assume $0 \leqslant \mathfrak{s} < 1/2$ when only adapted normal forms appear, and $0 \leqslant \mathfrak{s} \leqslant 1/3$ if a non-adapted normal form appears. Since the case $\mathfrak{s} = 0$ follows directly by Plancherel, we may assume $\mathfrak{s} > 0$.

Our assumptions in this case are that the Hessian determinant $\mathcal{H}_\phi$ does not vanish of infinite order anywhere (i.e., condition (H2) is satisfied). According to Subsection 4.1.2 we may restrict our attention to the localized measure

$$\langle \mu_{0,v}, f\rangle = \int_{\mathbb{R}^2\setminus\{0\}} f(x, \phi_v(x - v))\, \eta_v(x)\, |\mathcal{H}_\phi(x)|^{\mathfrak{s}}\, \mathrm{d}x,$$

where $v = (v_1, v_2)$ satisfies $v_1 \sim 1$, and either $v_2 = 0$ or $v_2 \sim 1$, and where η_v is a smooth nonnegative function with support in a small neighbourhood of v.

After changing to y or w coordinates from Section 4.2 we get that $\mu_{0,v}$ can be rewritten as

$$\langle \nu, f \rangle = \int f(x, \phi_{\mathrm{loc}}(x))\, a(x)\, |\mathcal{H}_{\phi_{\mathrm{loc}}}(x)|^{\mathfrak{s}}\, \mathrm{d}x,$$

where now a is smooth, nonnegative, and supported in a small neighbourhood of the origin, and where we have for ϕ_{loc} the normal form cases (i)-(vi) from Proposition 1.4.4. Recall that since we assume (H2), in case (i) of Proposition 1.4.4 the function φ vanishes identically.

The strategy will be to appropriately localize and rescale the problem, and then to use the associated "R^*R" operator. Let us begin by proving modifications of two essentially known results.

Lemma 4.3.1. *Let $\phi : \Omega \to \mathbb{R}$ be a smooth function on an open set $\Omega \subseteq \mathbb{R}^2$ contained in a ball of radius $\lesssim 1$, and let $\mathcal{H}_\phi = \partial_1^2\phi\partial_2^2\phi - (\partial_1\partial_2\phi)^2$ denote the Hessian determinant of ϕ. We consider the measure defined by*

$$\langle \mu, f \rangle := \int f(x_1, x_2, \phi(x))a(x)\mathrm{d}x,$$

where $a \in C_c^\infty(\Omega)$ satisfies $\|\partial^\tau a\|_{L^\infty(\Omega)} \lesssim_\tau 1$ for all multiindices τ. If we assume that on Ω we have $|\partial_1^2\phi| \sim 1$, $|\partial^\tau\phi| \lesssim_\tau 1$ for all multiindices τ, and that $|\mathcal{H}_\phi| \sim \varepsilon$ for a bounded, strictly positive (but possibly small) constant ε, then

$$|\widehat{\mu}(\xi)| \lesssim \varepsilon^{-1/2}(1 + |\xi|)^{-1}.$$

The claim also holds if ϕ and a depend on ε, assuming that the implicit constants appearing in the lemma can be taken to be independent of ε.

Proof. By compactness and translating we may assume that a is supported on a small neighbourhood of the origin. We also assume for simplicity that $|\partial_1\phi| \sim 1$, which can be achieved by applying a linear transformation to μ. The Fourier transform of μ is by definition

$$\widehat{\mu}(\xi) = \int e^{-i\Phi(x,\xi)}a(x)\mathrm{d}x,$$

where the phase function is of the form

$$\Phi(x, \xi) = x_1\xi_1 + x_2\xi_2 + \phi(x)\xi_3,$$

from which one easily sees that unless $|\xi_1| \sim |\xi_3| \gtrsim |\xi_2|$, we have a very fast decay. Let us denote

$$s_1 = \frac{\xi_1}{\xi_3}, \quad s_2 = \frac{\xi_2}{\xi_3}, \quad \lambda = \xi_3,$$

and rewrite the phase as

$$\Phi(x, \xi) = \lambda(s_1x_1 + s_2x_2 + \phi(x)),$$

where now $|s_1| \sim 1$ and $|s_2| \lesssim 1$.

Now either the x_1 derivative of Φ has no zeros on the domain of integration (e.g. when s_1 and $\partial_1\phi(0)$ are of the same sign), in which case we get a fast decay by integrating by parts, or there is a unique zero $x_1^c = x_1^c(x_2; s_1, s_2)$, depending smoothly on $(x_2; s_1, s_2)$ by the implicit function theorem, i.e., we have the relation

$$s_1 + (\partial_1\phi)(x_1^c, x_2) = 0. \tag{4.3.1}$$

In this case we apply the stationary phase method and get that

$$\widehat{\mu}(\xi) = \lambda^{-1/2}\int e^{-i\lambda\Psi(x_2;s_1,s_2)}a(x_2, s_1, s_2; \lambda)\mathrm{d}x_2,$$

where a is a smooth function in (x_2, s_1, s_2) and a classical symbol of order 0 in λ, and where $\Psi(x_2; s_1, s_2) := s_1x_1^c + s_2x_2 + \phi(x_1^c, x_2) = \lambda^{-1}\Phi(x_1^c, x_2, \xi)$.

Taking the x_2 derivative of (4.3.1) we get that

$$\partial_{x_2}x_1^c(x_2; s_1, s_2) = -\frac{\partial_1\partial_2\phi(x_1^c, x_2)}{\partial_1^2\phi(x_1^c, x_2)},$$

and the x_2 derivative of the new phase is by (4.3.1):

$$\begin{aligned}\lambda\partial_{x_2}\Psi(x_2; s_1, s_2) &= \lambda(s_1\partial_{x_2}x_1^c + s_2 + \partial_{x_2}x_1^c\,\partial_1\phi(x_1^c, x_2) + \partial_2\phi(x_1^c, x_2))\\ &= \lambda(s_2 + \partial_2\phi(x_1^c, x_2)).\end{aligned}$$

From this and the expression for $(x_1^c)'$ it follows that

$$\lambda\partial_{x_2}^2\Psi(x_2; s_1, s_2) = \lambda\frac{\mathcal{H}_\phi(x_1^c, x_2)}{\partial_1^2\phi(x_1^c, x_2)} \sim \lambda\varepsilon.$$

Thus, we may apply the van der Corput lemma 2.2.1, which then delivers the claim of the lemma. $\square$

The following lemma for obtaining mixed norm Fourier restriction estimates goes back essentially to Ginibre and Velo [38] (see also [56]). Compare with Lemma 2.3.1.

Lemma 4.3.2. *Assume that we are given a bounded open set Ω and functions $\Phi \in C^\infty(\Omega; \mathbb{R}^2)$, $\phi \in C^\infty(\Omega; \mathbb{R})$, $a \in L^\infty(\Omega)$. Let us consider the measure*

$$\langle\mu, f\rangle := \int f(\Phi(x), \phi(x))a(x)\mathrm{d}x$$

and the operator $T : f \mapsto f * \widehat{\mu}$. If Φ is injective and its Jacobian is of size $|J_\Phi| \sim A_1$, then the operator $L^1_{x_3}(\mathbb{R}; L^2_{(x_1,x_2)}(\mathbb{R}^2)) \to L^\infty_{x_3}(\mathbb{R}; L^2_{(x_1,x_2)}(\mathbb{R}^2))$ norm of T is bounded (up to a universal constant) by

$$A_1^{-1}\|a\|_{L^\infty}.$$

If one has furthermore the estimate

$$|\widehat{\mu}(\xi)| \leqslant A_2(1 + |\xi_3|)^{-1},$$

then for any $\mathfrak{s} \in [0, 1/2)$ *and* $(1/p_1', 1/p_3') = (1/2 - \mathfrak{s}, \mathfrak{s})$ *the operator* $L^{p_3}_{x_3}(\mathbb{R}; L^{p_1}_{(x_1,x_2)}(\mathbb{R}^2)) \to L^{p_3'}_{x_3}(\mathbb{R}; L^{p_1'}_{(x_1,x_2)}(\mathbb{R}^2))$ *norm of* T *is bounded (up to a constant depending on* $\mathfrak{s}$*) by*

$$(A_1^{-1}\|a\|_{L^\infty})^{1-2\mathfrak{s}} A_2^{2\mathfrak{s}}.$$

Proof. For functions on $\mathbb{R}^3$ let us denote by $\mathcal{F}'$ the inverse Fourier transform in the first two variables. Then it suffices for the first claim to prove that the L^∞ norm of $\mathcal{F}'\,\widehat{\mu}$ is bounded by $A_1^{-1}\|a\|_{L^\infty}$. But this follows immediately since $\mathcal{F}'\,\widehat{\mu}$ is equal by Fourier inversion to the Fourier transform of μ in the third coordinate, which is easily seen to be the function (up to a universal constant)

$$(x, \xi_3) \mapsto e^{-i\xi_3 \phi\circ\Phi^{-1}(x)}\, a \circ \Phi^{-1}(x) |J_\Phi(x)|^{-1}.$$

For the second claim we introduce the operator $T_{\xi_3} g := g * \widehat{\mu}(\cdot, \xi_3)$ defined for functions g on $\mathbb{R}^2$ and a fixed $\xi_3 \in \mathbb{R}$. Then the $L^1(\mathbb{R}^2) \to L^\infty(\mathbb{R}^2)$ norm of T_{ξ_3} is bounded by $A_2(1 + |\xi_3|)^{-1}$, and the $L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ norm is bounded up to a universal constant by $A_1^{-1}\|a\|_{L^\infty}$. Interpolating one gets that the $L^{p_1}(\mathbb{R}^2) \to L^{p_1'}(\mathbb{R}^2)$ norm is bounded by

$$(A_1^{-1}\|a\|_{L^\infty})^{1-2\mathfrak{s}} A_2^{2\mathfrak{s}} (1 + |\xi_3|)^{-2\mathfrak{s}}$$

for $p_1' = (1/2 - \mathfrak{s})$ and $\mathfrak{s} \in [0, 1/2]$. If one now writes a function f on $\mathbb{R}^3$ as $f(\xi_1, \xi_2, \xi_3) = f(\xi', \xi_3) = f_{\xi_3}(\xi')$, then

$$Tf(\xi', \xi_3) = \int (f_{\eta_3 - \xi_3} * \widehat{\mu}(\cdot, \eta_3))(\xi') \mathrm{d}\eta_3 = \int T_{\eta_3} f_{\eta_3 - \xi_3} \mathrm{d}\eta_3,$$

and so the claim follows by the (weak) Young inequality for $\mathfrak{s} < 1/2$. □

4.3.1 Normal form (i)

In this case the local form of the phase is

$$\phi_{\mathrm{loc}}(x) = x_2^k r(x),$$

where $r(0) \neq 0$ and the Hessian determinant vanishes of order $2k + k_0 - 2$ for some $k_0 \geqslant 0$, i.e., it has the normal form

$$\mathcal{H}_{\phi_{\mathrm{loc}}}(x) = x_2^{2k+k_0-2} q(x)$$

for some smooth function q satisfying $q(0) \neq 0$.

We begin by a dyadic decomposition $\nu = \sum_{j \gg 1} \nu_j$ in x_2 followed by scaling $x_2 \mapsto 2^{-j} x_2$. Namely, for a $j \gg 1$ we define

$$\langle \nu_j, f \rangle = \int f(x, \phi_{\mathrm{loc}}(x))\, a(x)\, \chi_1(2^j x_2)\, |\mathcal{H}_{\phi_{\mathrm{loc}}}(x)|^{\mathfrak{s}}\, \mathrm{d}x,$$

where $\chi_1(x_2)$ is supported where $|x_2| \sim 1$ and is such that $\sum_{j\in\mathbb{Z}} \chi_1(x_2) = 1$. Thus, by a Littlewood-Paley argument it suffices to prove

$$\|\widehat{f}\|^2_{L^2(\mathrm{d}\nu_j)} \lesssim \|f\|^2_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})},$$

with the implicit constant independent of j. Rescaling, this is equivalent to

$$\|\widehat{f}\|^2_{L^2(\mathrm{d}\tilde{\nu}_j)} \lesssim 2^{\mathfrak{s}jk_0}\|f\|^2_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})}, \tag{4.3.2}$$

where now

$$\langle \tilde{\nu}_j, f\rangle = \int f(x, \tilde{\phi}(x, 2^{-j}))\, a(x, 2^{-j})\mathrm{d}x.$$

The amplitude $a(x, 2^{-j})$ is now supported so that $|x_1| \ll 1$ and $|x_2| \sim 1$, and it is C^∞ having derivatives uniformly bounded. The phase is

$$\begin{aligned}\tilde{\phi}(x, 2^{-j}) &= 2^{jk}\phi_{\mathrm{loc}}(x_1, 2^{-j}x_2)\\ &= x_2^k r(x_1, 2^{-j}x_2).\end{aligned}$$

From this we have $|\partial_2\tilde{\phi}| \sim 1 \sim |\partial_2^2\tilde{\phi}|$ and one easily gets by using the definition of the Hessian determinant that

$$\begin{aligned}\mathcal{H}_{\tilde{\phi}}(x, 2^{-j}) &= 2^{j(2k-2)}\mathcal{H}_{\phi_{\mathrm{loc}}}(x_1, 2^{-j}x_2)\\ &= 2^{-jk_0}x_2^{2k+k_0-2}q(x_1, 2^{-j}x_2).\end{aligned}$$

Thus $|\mathcal{H}_{\tilde{\phi}}(x, 2^{-j})| \sim 2^{-jk_0}$, from which the estimate (4.3.2) follows by an application of Lemma 4.3.1 and subsequently Lemma 4.3.2.

4.3.2 Preliminary rescaling for cases (ii)-(vi)

In normal form cases (ii)-(vi) the principal face of $\mathcal{N}(\phi_{\mathrm{loc}})$ is compact (for the definition of the Newton polyhedron $\mathcal{N}(\phi_{\mathrm{loc}})$ of a smooth phase function ϕ_{loc} see Section 1.2), and so we use the scaling associated to it:

$$\delta_r^{\tilde{\kappa}}(x) = (r^{\tilde{\kappa}_1}x_1, r^{\tilde{\kappa}_2}x_2),$$

where in cases (ii)-(v) we have $\tilde{\kappa} = (1/2, 1/k)$ and in case (vi) we have $\tilde{\kappa} = (1/(2k), 1/k)$. In particular, for $j \gg 1$ we define

$$\langle \nu_j, f\rangle = \int f(x, \phi_{\mathrm{loc}}(x))\, a(x)\, \eta(\delta^{\tilde{\kappa}}_{2^j}x)\, |\mathcal{H}_{\phi_{\mathrm{loc}}}(x)|^{\mathfrak{s}}\, \mathrm{d}x,$$

where η is supported on an annulus and is such that $\sum_{j\in\mathbb{Z}}\eta(\delta^{\tilde{\kappa}}_{2^j}x) = 1$. By using Littlewood-Paley theory we get that it is sufficient to prove

$$\|\widehat{f}\|^2_{L^2(\mathrm{d}\nu_j)} \lesssim \|f\|^2_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})}.$$

Rescaling, the above estimate is equivalent to

$$\|\widehat{f}\|^2_{L^2(\mathrm{d}\tilde{\nu}_j)} \lesssim \|f\|^2_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})}, \tag{4.3.3}$$

where

$$\langle \tilde{\nu}_j, f\rangle = \int f(x, \tilde{\phi}(x,\delta))\, |\mathcal{H}_{\tilde{\phi}}(x,\delta)|^{\mathfrak{s}}\, a(x,\delta)\mathrm{d}x. \tag{4.3.4}$$

Here the amplitude $a(x,\delta)$ is supported on a fixed annulus around the origin,

$$\delta = (\delta_0, \delta_1, \delta_2) := (2^{-j\frac{k-1}{k}}, 2^{-j/2}, 2^{-j/k}) \tag{4.3.5}$$

in cases (ii)-(v), and

$$\delta = (\delta_1, \delta_2) := (2^{-j/(2k)}, 2^{-j/k})$$

in case (vi). The phase which one obtains in (4.3.4) is

$$\tilde{\phi}(x,\delta) := 2^j \phi_{\mathrm{loc}}(\delta_1 x_1, \delta_2 x_2).$$

The quantity δ_0 will be appear only later when we use the explicit normal forms. From the above phase form it follows that

$$\mathcal{H}_{\tilde{\phi}}(x,\delta) = 2^{j(k-2)/k} \mathcal{H}_{\phi_{\mathrm{loc}}}(\delta_1 x_1, \delta_2 x_2)$$

in cases (ii)-(v), and

$$\mathcal{H}_{\tilde{\phi}}(x,\delta) = 2^{j(2k-3)/k} \mathcal{H}_{\phi_{\mathrm{loc}}}(\delta_1 x_1, \delta_2 x_2)$$

in case (vi).

4.3.3 Normal forms (ii) and (iii)

Using the normal forms for ϕ_{loc} one gets in these cases

$$\begin{aligned}\tilde{\phi}(x,\delta) &= x_1^2 r_1(\delta_1 x_1, \delta_2 x_2) + x_2^k r_2(\delta_1 x_1, \delta_2 x_2),\\ \mathcal{H}_{\tilde{\phi}}(x,\delta) &= x_2^{k-2} q(\delta_1 x_1, \delta_2 x_2),\end{aligned}$$

where $r_1(0), r_2(0), q(0) \neq 0$, and $k \geqslant 3$. Hence, for the part where $|x_2| \gtrsim 1$ in (4.3.4) the Hessian is nondegenerate, and so we may localize to $|x_1| \sim 1$ and $|x_2| \ll 1$, and subsequently perform a dyadic decomposition in the x_2 coordinate, i.e., we define

$$\begin{aligned}\langle \nu_l, f\rangle &:= \int f(x, \tilde{\phi}(x,\delta))\, |x_2|^{\mathfrak{s}(k-2)}\, \chi_1(2^l x_2)\, a(x,\delta)\mathrm{d}x\\ &= 2^{-l-l\mathfrak{s}(k-2)} \int f(x_1, 2^{-l}x_2, \tilde{\phi}(x_1, 2^{-l}x_2, \delta))\, a(x,\delta,2^{-l})\mathrm{d}x,\end{aligned}$$

where now the amplitude is supported in a domain where $|x_1| \sim 1 \sim |x_2|$, and has uniformly bounded C^N norm for any N. Applying the Littlewood-Paley theorem again and rescaling, it is sufficient for us to prove

$$\|\widehat{f}\|^2_{L^2(\mathrm{d}\tilde{\nu}_{j,l})} \lesssim 2^{kl\mathfrak{s}} \|f\|^2_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})}, \tag{4.3.6}$$

where the rescaled measure is

$$\langle \tilde{\nu}_{j,l}, f\rangle = \int f(x, \tilde{\phi}(x_1, 2^{-l}x_2, \delta))\, a(x, \delta, 2^{-l})\mathrm{d}x.$$

The phase has now the form

$$x_1^2 r_1(\delta_1 x_1, \delta_2 x_2) + 2^{-kl} x_2^k r_2(\delta_1 x_1, 2^{-l}\delta_2 x_2) \tag{4.3.7}$$

on the domain $|x_1| \sim 1$ and $|x_2| \sim 1$, and its Hessian determinant is of size 2^{-kl}. By Lemma 4.3.1 we have

$$|\widehat{\tilde{\nu}}_{j,l}(\xi)| \lesssim 2^{kl/2}(1 + |\xi|)^{-1}.$$

And so the estimate (4.3.6) follows by Lemma 4.3.2.

4.3.4 Normal form (iv)

In this case we get

$$\begin{aligned}\tilde{\phi}(x, \delta) &= x_1^2 r_1(\delta_1 x_1) + (x_2 - \delta_0 x_1^2 \psi(\delta_1 x_1))^k r_2(\delta_1 x_1, \delta_2 x_2),\\ \mathcal{H}_{\tilde{\phi}}(x, \delta) &= (x_2 - \delta_0 x_1^2 \psi(\delta_1 x_1))^{k-2} q(\delta_1 x_1, \delta_2 x_2),\end{aligned}$$

where $r_1(0), r_2(0), q(0), \psi(0) \neq 0$, and $k \geqslant 3$. Therefore again, if $|x_2| \gtrsim 1$ the Hessian is nondegenerate and therefore we may concentrate on $|x_1| \sim 1$ and $|x_2| \ll 1$ in (4.3.4). We perform a dyadic decomposition, though this time depending on how close we are to the root of the Hessian determinant, i.e., we define

$$\langle \nu_l, f\rangle := \int f(x, \tilde{\phi}(x, \delta))\, |x_2 - \delta_0 x_1^2 \psi(\delta_1 x_1)|^{\mathfrak{s}(k-2)}\, \chi_1(2^l(x_2 - \delta_0 x_1^2 \psi(\delta_1 x_1)))\, a(x, \delta)\mathrm{d}x.$$

Next, after changing coordinates from x_2 to $x_2 + \delta_0 x_1^2 \psi(\delta_1 x_1)$ we may write

$$\langle \nu_l, f\rangle = \int f(x_1, x_2 + \delta_0 x_1^2 \psi(\delta_1 x_1), \phi_1(x, \delta))\, |x_2|^{\mathfrak{s}(k-2)}\, \chi_1(2^l x_2)\, a_1(x, \delta)\, \mathrm{d}x, \tag{4.3.8}$$

where

$$\begin{aligned}\phi_1(x, \delta) &= x_1^2 r_1(\delta_1 x_1) + x_2^k r_2(\delta_1 x_1, \delta_2 x_2 + \delta_0 \delta_2 x_1^2 \psi(\delta_1 x_1, \delta_2 x_2))\\ &= x_1^2 r_1(\delta_1 x_1) + x_2^k r_2(\delta_1 x_1, \delta_2 x_2 + (\delta_1 x_1)^2 \psi(\delta_1 x_1, \delta_2 x_2))\\ &= x_1^2 r_1(\delta_1 x_1) + x_2^k \tilde{r}_2(\delta_1 x_1, \delta_2 x_2).\end{aligned}$$

The function $\tilde{r}_2$ is a smooth and nonzero at the origin. Finally, we rescale in x_2 as $x_2 \mapsto 2^{-l} x_2$ and may write

$$\begin{aligned}\langle \nu_l, f\rangle = 2^{-l-l\mathfrak{s}(k-2)} \int & f(x_1, 2^{-l}x_2 + \delta_0 x_1^2 \psi(\delta_1 x_1), \phi_{j,l}(x, \delta, 2^{-l}))\\ & \chi_1(x_1)\, \chi_1(x_2)\, a(x, \delta, 2^{-l})\, \mathrm{d}x,\end{aligned} \tag{4.3.9}$$

where the amplitude is a smooth function and the phase is

$$\phi_{j,l}(x,\delta) = x_1^2 r_1(\delta_1 x_1) + 2^{-kl} x_2^k \tilde{r}_2(\delta_1 x_1, 2^{-l}\delta_2 x_2).$$

In order to obtain the estimate (4.3.3) we shall need essentially a variant of Lemma 4.3.2. Namely, we shall consider the analytic family of operators T_ζ defined by convolution against the Fourier transform of the measure

$$\mu_\zeta := \sum_{2^l \gg 1} 2^{l\mathfrak{s}(k-2)} 2^{-l\zeta(k-2)} \nu_l, \tag{4.3.10}$$

where ζ has real part between 0 and 1/2, and in particular, for a fixed $\xi_3 \in \mathbb{R}^3$, we shall consider the operator $T_\zeta^{\xi_3} : f \mapsto f * \widehat{\mu}_\zeta(\cdot, \xi_3)$. Note that we are interested in $\mu_{\mathfrak{s}}$ since this is precisely the sum of measures ν_l.

When the real part of ζ is 0 (i.e., $\zeta = it$, $t \in \mathbb{R}$) one considers the $L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ estimate for which we use the equation (4.3.8). In (4.3.8) we see that the amplitude is of size $2^{-l\mathfrak{s}(k-2)}$, which is precisely what we need in (4.3.10). Since the supports are disjoint when varying l, we get by a similar argumentation as in Lemma 4.3.2 that the operator $L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ norm of $T_{it}^{\xi_3}$ is $\lesssim 1$ (uniform in ξ_3 and t).

When the real part of ζ is 1/2 we need to prove

$$|\widehat{\mu}_{1/2+it}(\xi)| \lesssim (1 + |\xi_3|)^{-1} \tag{4.3.11}$$

with implicit constant independent of t and ξ_3, since this would give us that the operator norm of $T_{1/2+it}^{\xi_3}$ for mapping $L^1(\mathbb{R}^2) \to L^\infty(\mathbb{R}^2)$ is bounded by $(1 + |\xi_3|)^{-1}$.

Thus, under the assumption that we have the estimate (4.3.11) we may apply complex interpolation for each fixed ξ_3 to the analytic family of operators $T_\zeta^{\xi_3}$ and obtain that the operator norm of $T_{\mathfrak{s}}^{\xi_3}$ between spaces $L^{\mathfrak{p}_1}(\mathbb{R}^2) \to L^{\mathfrak{p}_1'}(\mathbb{R}^2)$ is $\lesssim (1 + |\xi_3|)^{-2\mathfrak{s}}$, and so in the same way as in the proof of Lemma 4.3.2 the (weak) Young inequality in the x_3 direction implies (4.3.3).

In proving (4.3.11) it suffices to show that

$$\sum_{2^l \gg 1} 2^{-l(1/2-\mathfrak{s})(k-2)} |\widehat{\nu}_l(\xi)| \lesssim (1 + |\xi_3|)^{-1}$$

for all $\xi \in \mathbb{R}^3$. By (4.3.9) the Fourier transform of a summand is

$$2^{-l(1/2-\mathfrak{s})(k-2)} \widehat{\nu}_l(\xi) = 2^{-kl/2} \int e^{-i\Phi(x,\xi,\delta,2^{-l})} \chi_1(x_1)\, \chi_1(x_2)\, a(x,\delta,2^{-l}) \mathrm{d}x,$$

where the phase function is

$$\begin{aligned} \Phi(x,\xi,\delta,2^{-l}) := {} & \xi_1 x_1 + \xi_2 \delta_0 x_1^2 \psi(\delta_1 x_1) + \xi_3 x_1^2 r_1(\delta_1 x_1) \\ & + 2^{-l}\xi_2 x_2 + 2^{-kl} \xi_3 x_2^k \tilde{r}_2(\delta_1 x_1, 2^{-l}\delta_2 x_2). \end{aligned}$$

We see that when either $|\xi_1| \gg \max\{|\xi_2|, |\xi_3|\}$ or $|\xi_3| \gg \max\{|\xi_1|, |\xi_2|\}$ we can use integration by parts in the x_1 variable and get a very fast decay. This is also the case when $|\xi_1| \sim |\xi_2|$ are much greater than $|\xi_3|$, or when $|\xi_2| \sim |\xi_3|$ are much greater than $|\xi_1|$. If we have $|\xi_2| \gtrsim |\xi_3|$, then we may use integration by parts in x_2 and get

$$|2^{-l(1/2-\mathfrak{s})(k-2)}\widehat{\nu}_l(\xi)| \lesssim 2^{-kl/2}(1+2^{-l}|\xi_2|)^{-1} \lesssim 2^{-kl/2}(1+2^{-l}|\xi_3|)^{-1},$$

from which (4.3.11) follows since $k \geqslant 3$. We are thus left with the case when $|\xi_1| \sim |\xi_3| \gg |\xi_2|$.

Case 1. $2^{-kl}|\xi_3| \lesssim 1$**.** Here we use the van der Corput lemma 2.2.1 in x_1 only and get

$$|2^{-l(1/2-\mathfrak{s})(k-2)}\widehat{\nu}_l(\xi)| \lesssim 2^{-kl/2}|\xi_3|^{-1/2}.$$

Summation in l then gives precisely (4.3.11).

Case 2. $2^{-l}|\xi_2| \not\sim 2^{-kl}|\xi_3|$ **and** $2^{-kl}|\xi_3| \gg 1$**.** We may use in this case integration by parts in x_2 and then the van der Corput lemma 2.2.1 in x_1 and get

$$\begin{aligned}|2^{-l(1/2-\mathfrak{s})(k-2)}\widehat{\nu}_l(\xi)| &\lesssim 2^{-kl/2}|\xi_3|^{-1/2}\,(2^{-kl}|\xi_3|)^{-1}\\ &\lesssim 2^{kl/2}|\xi_3|^{-3/2}.\end{aligned}$$

We may now sum in l.

Case 3. $2^{-l}|\xi_2| \sim 2^{-kl}|\xi_3| \gg 1$**.** Here we have by iterative stationary phase (first in x_2 and then in x_1) that

$$|2^{-l(1/2-\mathfrak{s})(k-2)}\widehat{\nu}_l(\xi)| \lesssim 2^{-kl/2}|\xi_3|^{-1/2}\,(2^{-kl}|\xi_3|)^{-1/2} = |\xi_3|^{-1}.$$

Here we note that $2^{l(k-1)} \sim |\xi_3|\,|\xi_2|^{-1}$, and so we sum only over finitely many (i.e., $\mathcal{O}(1)$) l for each fixed ξ. Thus, here we also have the estimate (4.3.11).

4.3.5 Normal form (v)

Recall that here

$$\begin{aligned}\phi_{\mathrm{loc}}(x) &= x_1^2 r_1(x) + (x_2 - x_1^2\psi(x_1))^k r_2(x),\\ \mathcal{H}_{\phi_{\mathrm{loc}}}(x) &= (x_2 - x_1^2\psi(x_1))^{k-2} q(x),\end{aligned}$$

where we know that $k \geqslant 3$, $r_1(0), r_2(0), q(0), \psi(0) \neq 0$. Furthermore, recall that this corresponded to the w coordinates when deriving the normal forms, and we have shown that we additionally have in this case:

$$\partial_2^{\tau_2} r_1(0) \neq 0, \qquad \text{for all } \tau_2 \in \{0, 1, \dots, k-1\}.$$

In fact, one has the relationship:

$$c\,\tau_2 \partial_2^{\tau_2-1} r_1(0) = \partial_2^{\tau_2} r_1(0), \qquad \text{for all } \tau_2 \in \{1, \dots, k-1\},$$

where c is some fixed nonzero constant (see Subsection 4.2.5). This implies for example the relation:

$$r_1(0)\,\partial_2^2 r_1(0) - 2(\partial_2 r_1)^2(0) = 0. \tag{4.3.12}$$

From the above normal form we have

$$\begin{aligned}\tilde{\phi}(x,\delta) &= x_1^2 r_1(\delta_1 x_1, \delta_2 x_2) + (x_2 - \delta_0 x_1^2 \psi(\delta_1 x_1))^k r_2(\delta_1 x_1, \delta_2 x_2),\\ \mathcal{H}_{\tilde{\phi}}(x,\delta) &= (x_2 - \delta_0 x_1^2 \psi(\delta_1 x_1))^{k-2} q(\delta_1 x_1, \delta_2 x_2).\end{aligned}$$

We may as usual localize to $|x_1| \sim 1$ and $|x_2| \ll 1$. We shall abuse the notation a bit and denote this localized measure again by $\tilde{\nu}_j$. After changing coordinates from x_2 to $x_2 + \delta_0 x_1^2 \psi(\delta_1 x_1)$ we may write

$$\langle \tilde{\nu}_j, f \rangle = \int f(x_1, x_2 + \delta_0 x_1^2 \psi(\delta_1 x_1), \phi_1(x,\delta))\, |x_2|^{\mathfrak{s}(k-2)}\, a_1(x,\delta)\, \chi_1(x_1)\, \chi_0(x_2)\, \mathrm{d}x$$

with the phase being

$$\phi_1(x,\delta) = x_1^2 \tilde{r}_1(\delta_1 x_1, \delta_2 x_2) + x_2^k \tilde{r}_2(\delta_1 x_1, \delta_2 x_2),$$

where $\tilde{r}_1, \tilde{r}_2$ are smooth functions, nonzero at the origin, and satisfy the same properties and relations as r_1 and r_2 mentioned at the beginning of this subsection. As in the case (iv), we also decompose the measure $\tilde{\nu}_j$ as $\tilde{\nu}_j = \sum_l \nu_l$, where

$$\langle \nu_l, f \rangle = \int f(x_1, x_2 + \delta_0 x_1^2 \psi(\delta_1 x_1), \phi_1(x,\delta))\, |x_2|^{\mathfrak{s}(k-2)}\, a_1(x,\delta)\, \chi_1(x_1)\, \chi_1(2^l x_2)\, \mathrm{d}x.$$

Next, we shall be interested in the rescaled phase:

$$\phi_l(x,\delta,2^{-l}) = \phi_1(x_1, 2^{-l}x_2, \delta) = \tilde{\phi}(x_1, 2^{-l}x_2 + \delta_0 x_1^2 \psi(\delta_1 x_1), \delta).$$

Now we need a relation between the Hessian determinant of ϕ_l and the Hessian determinant of $\tilde{\phi}$. For this let us denote for simplicity

$$\varphi(x_1,\delta_1) := \delta_1^2 x_1^2 \psi(\delta_1 x_1).$$

The reason why we have not included the factor δ_2^{-1} will be clear later (recall from (4.3.5) that $\delta_0 = \delta_1^2 \delta_2^{-1}$). A direct calculation shows then

$$\mathcal{H}_{\phi_l} = 2^{-2l}\mathcal{H}_{\tilde{\phi}} + \delta_2^{-1} 2^l \partial_1^2 \varphi \partial_2 \phi_l \partial_2^2 \phi_l, \tag{4.3.13}$$

and due to our localization we have $|\mathcal{H}_{\tilde{\phi}}| \sim 2^{-l(k-2)}$.

We use the same complex interpolation idea as in (iv) according to which it suffices to prove

$$\sum_{2^l \gg 1} 2^{-l(1/2-\mathfrak{s})(k-2)} |\widehat{\nu}_l(\xi)| \lesssim (1 + |\xi_3|)^{-1},$$

where after rescaling $x_2 \mapsto 2^{-l}x_2$ we have

$$2^{-l(1/2-\mathfrak{s})(k-2)} \widehat{\nu}_l(\xi) = 2^{-kl/2} \int e^{i\Phi_0(x,\xi,\delta,2^{-l})} a(x,\delta,2^{-l}) \mathrm{d}x,$$

where the phase function for the Fourier transform of ν_l is

$$\begin{aligned}\Phi_0(x,\xi,\delta,2^{-l}) &:= \xi_1 x_1 + \xi_2\delta_0 x_1^2\psi(\delta_1 x_1) + \xi_3 x_1^2\tilde{r}_1(\delta_1 x_1, 2^{-l}\delta_2 x_2)\\ &\quad + \xi_2 2^{-l}x_2 + \xi_3 2^{-kl}x_2^k\tilde{r}_2(\delta_1 x_1, 2^{-l}\delta_2 x_2)\\ &= \xi_1 x_1 + \xi_2\delta_2^{-1}\varphi(x_1,\delta_1) + \xi_2 2^{-l}x_2 + \xi_3\phi_l(x,\delta,2^{-l}).\end{aligned}$$

The amplitude localizes the integration to $|x_1| \sim 1 \sim |x_2|$.

Using the same argumentation as in the case (iv) we can reduce ourselves to the case when $|\xi_1| \sim |\xi_3|$, $|\xi_2| \ll |\xi_3|$, and $|\xi_3|2^{-kl} \gg 1$ are satisfied.

Now let us make some further reductions using the fact that $\partial_2\tilde{r}_1(0), \partial_2^2\tilde{r}_1(0) \neq 0$. The x_2 derivative of the phase Φ_0 contains three terms of respective sizes: $\sim |2^{-l}\delta_2\xi_3|$, $\sim |2^{-l}\xi_2|$, and $\sim |2^{-kl}\xi_3|$. If we may integrate by parts in x_2 (i.e., if one of the above terms is much larger than the other two), we can get an admissible estimate and sum in l. If $|2^{-kl}\xi_3|$ is comparable to the larger of the other two terms, then one easily sees that the second derivative in x_2 is necessarily of size $|2^{-kl}\xi_3|$, and so in this case we get by iterative stationary phase the estimate

$$2^{-l(1/2-\mathfrak{s})(k-2)}|\widehat{\nu}_l(\xi)| \lesssim (1+|\xi_3|)^{-1}.$$

Note that we do not need to sum in l since there are only finitely many l satisfying one of the relations $|2^{-kl}\xi_3| \sim |2^{-l}\delta_2\xi_3|$ or $|2^{-kl}\xi_3| \sim |2^{-l}\xi_2|$.

We are thus now reduced to the case when

$$|2^{-l}\xi_2| \sim |2^{-l}\delta_2\xi_3| \gg |2^{-kl}\xi_3|, \qquad |\xi_1| \sim |\xi_3|, \qquad \text{and} \qquad |\xi_3|2^{-kl} \gg 1.$$

At this point we introduce some further notation:

$$\lambda := \xi_3, \quad s_1 := \frac{\xi_1}{\xi_3}, \quad s_2 := \frac{\xi_2}{\delta_2\xi_3}, \quad \varepsilon := 2^{-l}\delta_2,$$

and so we have $|s_1| \sim 1 \sim |s_2|$, $\lambda 2^{-kl} \gg 1$, and $\varepsilon \gg 2^{-kl}$. The phase Φ_0 can now be rewritten as $\lambda\Phi$, where Φ is

$$\Phi(x,s_1,s_2,\delta_1,\varepsilon,2^{-kl}) = s_1x_1 + s_2\delta_1^2x_1^2\psi(\delta_1x_1) + s_2\varepsilon x_2 + \phi_l(x,\delta,2^{-l}),$$

since we note from the form of ϕ_l that ϕ_l can also be taken to depend on $(x_1,x_2,\delta_1,\varepsilon,2^{-kl})$.

Let us now apply the stationary phase method in x_1. We may rewrite the phase as

$$\Phi(x,s_1,s_2,\delta_1,\varepsilon,2^{-kl}) = s_1x_1 + s_2\varphi + s_2\varepsilon x_2 + \phi_l,$$

where we recall that $\varphi(x_1,\delta) = \delta_1^2x_1^2\psi(\delta_1x_1)$. We may assume that there is a stationary point for the x_1 derivative since $|\partial_1^2\phi_l| \sim 1$ and $|s_1| \sim 1$, and as otherwise we may use integration by parts.

We denote by $x_1^c = x_1^c(x_2,s_1,s_2,\delta_1,\varepsilon,2^{-kl})$ the function such that

$$(\partial_1\Phi)(x_1^c,x_2,s_1,s_2,\delta_1,\varepsilon,2^{-kl}) = s_1 + s_2\partial_1\varphi + \partial_1\phi_l = 0. \tag{4.3.14}$$

Taking the x_2 derivative we get

$$s_2(x_1^c)'\partial_1^2\varphi + (x_1^c)'\partial_1^2\phi_l + \partial_1\partial_2\phi_l = 0. \tag{4.3.15}$$

After applying the stationary phase method in x_1 we gain a decay factor of $\lambda^{-1/2}$, i.e., we have

$$2^{-l(1/2-\mathfrak{s})(k-2)}\widehat{\nu_l}(\xi) = \lambda^{-1/2}2^{-kl/2}\int e^{-i\lambda\tilde{\Phi}(x_2,s_1,s_2,\delta_1,\varepsilon,2^{-kl})}a(x_2,s_1,s_2,\delta,2^{-l};\lambda)\mathrm{d}x_2,$$

where the new phase is

$$\tilde{\Phi}(x_2,s_1,s_2,\delta_1,\varepsilon,2^{-kl}) = s_1x_1^c + s_2\varphi(x_1^c,\delta_1) + s_2\varepsilon x_2 + \phi_l(x_1^c,x_2,\delta,2^{-l}),$$

and the amplitude a is a classical symbol in λ of order 0.

Taking the x_2 derivative of the expression for the new phase $\tilde{\Phi}$ and using the equation (4.3.14) we get

$$\tilde{\Phi}' = s_2\varepsilon + \partial_2\phi_l. \tag{4.3.16}$$

Therefore, the second derivative of the new phase is

$$\begin{aligned}\tilde{\Phi}'' &= (\partial_2\phi_l)' \\ &= \partial_2^2\phi_l + (x_1^c)'\partial_1\partial_2\phi_l.\end{aligned} \tag{4.3.17}$$

Now using in order (4.3.15), the definition of $\mathcal{H}_{\phi_l}$ (4.3.13), (4.3.16), and (4.3.17), we obtain

$$\begin{aligned}(\partial_1^2\phi_l)\tilde{\Phi}'' &= \partial_1^2\phi_l\,\partial_2^2\phi_l + \partial_1\partial_2\phi_l(-\partial_1\partial_2\phi_l - s_2(x_1^c)'\partial_1^2\varphi) \\ &= \mathcal{H}_{\phi_l} - s_2(x_1^c)'\partial_1^2\varphi\,\partial_1\partial_2\phi_l \\ &= 2^{-2l}\mathcal{H}_{\tilde{\phi}} + \delta_2^{-1}2^l\partial_1^2\varphi\partial_2\phi_l\partial_2^2\phi_l - s_2(x_1^c)'\partial_1^2\varphi\,\partial_1\partial_2\phi_l \\ &= 2^{-2l}\mathcal{H}_{\tilde{\phi}} + \varepsilon^{-1}\partial_1^2\varphi\,\partial_2^2\phi_l(\tilde{\Phi}' - \varepsilon s_2) - s_2(x_1^c)'\partial_1^2\varphi\,\partial_1\partial_2\phi_l \\ &= 2^{-2l}\mathcal{H}_{\tilde{\phi}} - s_2\partial_1^2\varphi\,\partial_2^2\phi_l - s_2(x_1^c)'\partial_1^2\varphi\,\partial_1\partial_2\phi_l + \varepsilon^{-1}\partial_1^2\varphi\,\partial_2^2\phi_l\,\tilde{\Phi}' \\ &= 2^{-2l}\mathcal{H}_{\tilde{\phi}} - s_2\partial_1^2\varphi\,\tilde{\Phi}'' + \varepsilon^{-1}\partial_1^2\varphi\,\partial_2^2\phi_l\,\tilde{\Phi}'.\end{aligned}$$

Thus, we get

$$(s_2\partial_1^2\varphi + \partial_1^2\phi_l)\tilde{\Phi}'' = 2^{-2l}\mathcal{H}_{\tilde{\phi}} + \varepsilon^{-1}\partial_1^2\varphi\,\partial_2^2\phi_l\,\tilde{\Phi}'. \tag{4.3.18}$$

Note that we have $|\varepsilon^{-1}\partial_1^2\varphi\,\partial_2^2\phi_l| \ll \delta_1^2 \ll 1$ and $|s_2\partial_1^2\varphi + \partial_1^2\phi_l| \sim 1$, and recall that $|2^{-2l}\mathcal{H}_{\tilde{\phi}}| \sim 2^{-kl}$. We claim that either $|\tilde{\Phi}'| \lesssim 2^{-kl}$ on the whole domain of integration (i.e., for $|x_2| \sim 1$), or that $|\tilde{\Phi}'| \gtrsim 2^{-kl}$ on the whole domain of integration. This can be shown by using the formula for the solution of a linear first order ODE (considering $\tilde{\Phi}'$ as the unknown), or by arguing by contradiction.

Let us argue by contradiction in the following way. Let us assume that there exists a point $|x_2^0| \sim 1$ such that $|\tilde{\Phi}'(x_2^0)| \leqslant 2^{-kl}$. Furthermore, let us assume that there exists a point $|x_2^1| \sim 1$ where one has $|\tilde{\Phi}'| = C_1 2^{-kl}$ for some sufficiently large C_1, and let us assume that x_2^1 is the closest point to x_2^0 satisfying this condition in the sense that $|\tilde{\Phi}'| < C_1 2^{-kl}$

between x_2^0 and x_2^1. Then the mean value theorem implies that there is a point between x_2^0 and x_2^1 where we have $|\tilde{\Phi}''| \geqslant C_2 2^{-kl}$, where C_2 can be taken to tend to ∞ as C_1 tends to ∞. On the other hand, the equation (4.3.18) implies that on the interval between x_2^0 and x_2^1 we have $|\tilde{\Phi}''| \leqslant C_3 2^{-kl}$, where we can take C_3 to be a fixed constant if δ_1 is taken to be sufficiently small when C_1 and C_2 are large (we can always take say C_1 of size δ_1^{-1}). This is a contradiction, i.e., the point x_2^1 where one has $|\tilde{\Phi}'| \geqslant C_1 2^{-kl}$ for a too large C_1 cannot exist within the integration domain.

Now in the case that $|\tilde{\Phi}'| \gtrsim 2^{-kl}$ we may apply integration by parts and get an estimate summable in l. Let us therefore assume $|\tilde{\Phi}'| \lesssim 2^{-kl}$, in which case we have $|\tilde{\Phi}''| \sim 2^{-kl}$ by (4.3.18). Then the van der Corput lemma 2.2.1 implies that

$$2^{-l(1/2-\mathfrak{s})(k-2)} |\widehat{\nu_l}(\xi)| \lesssim (1 + |\xi_3|)^{-1}.$$

The problem is now that a priori we may not sum this estimate in l. Luckily, it turns out that one can pin down the size of 2^{-l}, which in turn will pin down the number l to a finite set of size $\mathcal{O}(1)$. In order to prove this we use the expression (4.3.16) and the normal form of ϕ_l:

$$\phi_l(x, \delta, 2^{-l}) = x_1^2 \tilde{r}_1(\delta_1 x_1, \varepsilon x_2) + 2^{-kl} x_2^k \tilde{r}_2(\delta_1 x_1, \varepsilon x_2),$$

from which one has

$$(\partial_2 \phi_l)(x, \delta, 2^{-l}) = \varepsilon x_1^2 (\partial_2 \tilde{r}_1)(\delta_1 x_1, \varepsilon x_2) + 2^{-kl} x_2^{k-1} \tilde{r}_3(\delta_1 x_1, \varepsilon x_2), \tag{4.3.19}$$

where $\tilde{r}_3(0) \neq 0$ is a smooth function.

The idea is as follows. First, by compactness we may assume that we integrate in x_2 over a sufficiently small neighbourhood of a point x_2^0 satisfying $|x_2^0| \sim 1$. In particular, we may write

$$\begin{aligned}\tilde{\Phi}'(x_2, s_1, s_2, \delta_1, \varepsilon, 2^{-kl}) &= \tilde{\Phi}'(x_2^0, s_1, s_2, \delta_1, \varepsilon, 2^{-kl}) + o(|\tilde{\Phi}''|) \\ &= \tilde{\Phi}'(x_2^0, s_1, s_2, \delta_1, \varepsilon, 2^{-kl}) + o(2^{-kl}).\end{aligned}$$

Thus, it suffices to prove that

$$|\tilde{\Phi}'(x_2^0, s_1, s_2, \delta_1, \varepsilon, 2^{-kl})| = |s_2 \varepsilon + \partial_2 \phi_l(x_1^c, x_2^0, s_1, s_2, \delta_1, \varepsilon, 2^{-kl})| \ll 2^{-kl}$$

can happen only for finitely many l. If the above inequality does not hold, then we may simply integrate by parts and are able to simply sum in l afterwards.

If we now develop both terms in $\partial_2 \phi_l$ in the ε and 2^{-kl} variables (recall that x_1^c depends on both ε and 2^{-kl}), then one gets that the expression for $\tilde{\Phi}'$ is of the form

$$s_2 \varepsilon + \sum_{i=1}^{k-1} \varepsilon^i f_i(x_2^0, s_1, s_2, \delta_1) + 2^{-kl} g_0(x_2^0, s_1, s_2, \delta_1) + o(2^{-kl}),$$

where we used the fact that $\varepsilon^k = (\delta_2 2^{-l})^k \ll 2^{-kl}$. Note that we have $|g_0| \sim 1$ by (4.3.19) (and also $|f_1| \sim 1$, but this is not important). We have to find out how many l's satisfy

$$\begin{aligned}\Big| \tilde{f}_1(x_2^0, s_1, s_2, \delta_1) &+ \sum_{i=2}^{k-1} \varepsilon^{i-1} f_i(x_2^0, s_1, s_2, \delta_1) \\ &+ \varepsilon^{-1} 2^{-kl} g_0(x_2^0, s_1, s_2, \delta_1) + o(\varepsilon^{-1} 2^{-kl}) \Big| \ll \varepsilon^{-1} 2^{-kl},\end{aligned}$$

where $\tilde{f}_1(x_2^0, s_1, s_2, \delta_1) := s_2 + f_1(x_2^0, s_1, s_2, \delta_1)$. But now one easily shows that this inequality is possible only if at least two of the terms are comparable in size (precisely because $|g_0| \sim 1$). This implies in particular that we can determine l in terms of $(x_2^0, s_1, s_2, \delta_1)$, which finishes the proof.

We mention that, interestingly, one can prove that $f_2(x_2^0, s_1, s_2, 0) = 0$, a consequence of the relation (4.3.12).

4.3.6 Normal form (vi)

Here we obtain

$$\begin{aligned}\tilde{\phi}(x,\delta) &= (x_2 - x_1^2\psi(\delta_1 x_1))^k r(\delta_1 x_1, \delta_2 x_2),\\ \mathcal{H}_{\tilde{\phi}}(x,\delta) &= (x_2 - x_1^2\psi(\delta_1 x_1))^{2k-3} q(\delta_1 x_1, \delta_2 x_2),\end{aligned}$$

where $r(0), q(0), \psi(0) \neq 0$. Thus, we may localize to the part where $|x_2 - x_1^2\psi(\delta_1 x_1)| \ll 1$, i.e., it is sufficient to consider the measure

$$f \mapsto \int f(x, \tilde{\phi}(x,\delta))\, |(x_2 - x_1^2\psi(\delta_1 x_1))^{2k-3} q(\delta_1 x_1, \delta_2 x_2)|^{\mathfrak{s}}\, \chi_0(\tilde{\phi}(x,\delta))\, a(x,\delta)\mathrm{d}x$$

since $|\tilde{\phi}(x,\delta)| \sim |x_2 - x_1^2\psi(\delta_1 x_1)|^k$. Note that here we have $|x_1| \sim 1 \sim |x_2|$.

Now, the next idea is to use, as in [51], a Littlewood-Paley decomposition in the x_3 direction (for the mixed norm Littlewood-Paley theory see [62]) and reduce ourselves to proving the Fourier restriction estimate for the measure piece

$$\begin{aligned}\langle \nu_l, f\rangle = \int & f(x, \tilde{\phi}(x,\delta))\, |(x_2 - x_1^2\psi(\delta_1 x_1))^{2k-3} q(\delta_1 x_1, \delta_2 x_2)|^{\mathfrak{s}}\\ & \chi_1(2^{kl}(\tilde{\phi}(x,\delta)))\, a(x,\delta)\mathrm{d}x.\end{aligned}$$

Using the coordinate transformation $x_2 \mapsto x_2 + x_1^2\psi(\delta_1 x_1)$ we may write

$$\begin{aligned}\langle \nu_l, f\rangle = \int & f(x_1, x_2 + x_1^2\psi(\delta_1 x_1), x_2^k\tilde{r}(\delta_1 x_1, \delta_2 x_2))\\ & \times |x_2^{2k-3}\tilde{q}(\delta_1 x_1, \delta_2 x_2)|^{\mathfrak{s}}\, \chi_1(2^{kl} x_2^k \tilde{r}(\delta_1 x_1, \delta_2 x_2))\, \tilde{a}(x,\delta)\mathrm{d}x,\end{aligned}$$

where $|\tilde{r}| \sim 1$ is a smooth function. Finally, we use the coordinate transformation $x_2 \mapsto 2^{-l}x_2$ and rescale f in the third coordinate. Then we are reduced to proving the Fourier restriction estimate

$$\|\widehat{f}\|^2_{L^2(\mathrm{d}\tilde{\nu}_{j,l})} \leqslant C 2^{l(1-3\mathfrak{s})} \|f\|^2_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})}, \tag{4.3.20}$$

for the measure

$$\langle \tilde{\nu}_{j,l}, f\rangle = \int f(x_1, 2^{-l}x_2 + x_1^2\psi(\delta_1 x_1), x_2^k\tilde{r}(\delta_1 x_1, 2^{-l}\delta_2 x_2)) a(x,\delta,2^{-l})\mathrm{d}x \tag{4.3.21}$$

where a is supported so that $|x_1| \sim 1$ and $|x_2| \sim 1$. Now we note that the estimate for $\mathfrak{s} = 0$ follows by Plancherel, while the estimate for $\mathfrak{s} = 1/3$ is going to be shown in

Section 4.4 since the form of the measure $\tilde{\nu}_{j,l}$ coincides with the form in (4.4.10) below. Interpolating, we obtain the estimate for all $0 \leqslant \mathfrak{s} \leqslant 1/3$.

Note that when $1/p_1' = 1/p_3' = 1/4$, then one can simplify the proof by a modification of Lemma 4.3.2, i.e., by using the Fourier decay of $\tilde{\nu}_{j,l}$, which is easily seen to be

$$|\widehat{\tilde{\nu}}_{j,l}(\xi)| \lesssim 2^{l/2}(1+|\xi|)^{-1},$$

and by using the Plancherel theorem, but this time in the (x_1, x_3)-plane (which is why it works only for $1/p_1' = 1/p_3'$) since the mapping $(x_1, x_2) \mapsto (x_1, x_2^k \tilde{r}(\delta_1 x_1, 2^{-l}\delta_2 x_2))$ has Jacobian of size ~ 1. In fact, in Section 4.4 we shall combine this idea of using Lemma 4.3.2 together with the methods used in [51] (and Chapter 3).

A Knapp-type example

Let us now show by using a Knapp-type example that one cannot get the estimate (4.3.20) for $\mathfrak{s} > 1/3$. Let us consider the function φ_ϵ defined by

$$\widehat{\varphi}_\epsilon(x) = \chi_1\Big(\frac{x_1}{\epsilon^\delta}\Big)\chi_1\Big(\frac{x_2}{\epsilon^{2\delta}}\Big)\chi_1\Big(\frac{x_3}{\epsilon}\Big)$$

for some small ϵ and δ. Its mixed L^p norm is

$$\|\varphi_\epsilon\|_{L^p(\mathbb{R}^3)} \sim \epsilon^{\frac{3\delta}{p_1'}+\frac{1}{p_3'}}.$$

Now, in the integral

$$\int |\widehat{\varphi}_\epsilon|^2 \mathrm{d}\nu = \int |\widehat{\varphi}_\epsilon|^2(x, \phi_{\mathrm{loc}}(x)) a(x) |\mathcal{H}_{\phi_{\mathrm{loc}}}(x)|^{\mathfrak{s}} \mathrm{d}x$$

we integrate over the set

$$D_\epsilon^0 := \{x \in \mathbb{R}^2 : |x_1| \lesssim \epsilon^\delta, \quad |x_2| \lesssim \epsilon^{2\delta}, \quad |\phi_{\mathrm{loc}}(x)| \sim |x_2 - x_1^2\psi(x_1)|^k \lesssim \epsilon\}$$

by definition of φ_ϵ. If δ is sufficiently small, D_ϵ^0 contains the set

$$D_\epsilon := \{x \in \mathbb{R}^2 : |x_1| \lesssim \epsilon^\delta, \quad |\phi_{\mathrm{loc}}(x)| \sim |x_2 - x_1^2\psi(x_1)| \lesssim \epsilon^{1/k}\},$$

and so if the Fourier restriction estimate holds, one has

$$\begin{aligned}
\epsilon^{\frac{6\delta}{p_1'}+\frac{2}{p_3'}} \sim \|\varphi_\epsilon\|^2_{L^p(\mathbb{R}^3)} \gtrsim \int |\widehat{\varphi}_\epsilon|^2 \mathrm{d}\nu &\gtrsim \int_{D_\epsilon} |x_2 - x_1^2\psi(x_1)|^{\mathfrak{s}(2k-3)} \mathrm{d}x \\
&\sim \epsilon^\delta \int_{|y| \lesssim \epsilon^{1/k}} |y|^{\mathfrak{s}(2k-3)} \mathrm{d}y \\
&\sim \epsilon^{\delta + (\mathfrak{s}(2k-3)+1)/k}.
\end{aligned}$$

Letting ϵ and then δ tend to 0 we obtain the condition

$$\frac{1}{p_3'} \leqslant \frac{\mathfrak{s}(2k-3)+1}{2k} = \mathfrak{s} + \frac{1-3\mathfrak{s}}{2k}.$$

Since we are interested in $1/p_1' = 1/2 - \mathfrak{s}$ and $1/p_3' = \mathfrak{s}$, the above inequality implies precisely $\mathfrak{s} \leqslant 1/3$.

4.4 Fourier restriction without a mitigating factor

Here we prove Theorem 1.4.2, i.e., the estimate

$$\|\widehat{f}\|_{L^2(\mathrm{d}\mu)} \leqslant C\|f\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})}$$

for μ the surface carried measure of the form

$$\langle \mu, f\rangle = \int_{\mathbb{R}^2\setminus\{0\}} f(x,\phi(x))\,|x|_\kappa^{\mathcal{D}_\mathcal{W}}\,\mathrm{d}x,$$

where

$$\mathcal{D}_\mathcal{W} = 2\Big(\frac{|\kappa|}{p_1'} + \frac{\mathcal{D}}{p_3'} - \frac{|\kappa|}{2}\Big).$$

Recall that this $\mathcal{D}_\mathcal{W}$ is chosen (depending on $(p_1,p_3) \in (1,2]^2$) precisely so that the above restriction estimate is equivalent to the local estimate

$$\|\widehat{f}\|_{L^2(\mathrm{d}\mu_0)} \leqslant C\|f\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})},$$

where μ_0 is the surface carried measure

$$\langle \mu_0, f\rangle = \int_{\mathbb{R}^2\setminus\{0\}} f(x,\phi(x))\,\eta(x)\,|x|_\kappa^{\mathcal{D}_\mathcal{W}}\,\mathrm{d}x \tag{4.4.1}$$

for $\eta \in C_c^\infty(\mathbb{R}^2\setminus\{0\})$ identically equal to 1 in an annulus.

Note that $|x|_\kappa^{\mathcal{D}_\mathcal{W}}$ is not smooth near the axes. Luckily, we shall be able to circumvent this problem by using the Littlewood-Paley theorem to localize away from the axes, as was done in the case with the mitigating factor.

Now we recall the necessary conditions from Proposition 3.1.1 obtained through the Knapp-type examples. Let us fix a point v such that $\eta(v) \neq 0$ and let η_v be a smooth cutoff function identically equal to η on a small neighbourhood of v. It suffices to consider the measure

$$\langle \mu_{0,v}, f\rangle = \int_{\mathbb{R}^2\setminus\{0\}} f(x,\phi_v(x-v))\,\eta_v(x)\,|x|_\kappa^{\mathcal{D}_\mathcal{W}}\,\mathrm{d}x, \tag{4.4.2}$$

where we recall from the introduction that

$$\phi_v(x) = \phi(x+v) - \phi(v) - x\cdot\nabla\phi(v).$$

We recall also that $h_{\mathrm{lin}}(\phi,v)$ is the linear height of ϕ_v at its origin, that $h(\phi,v)$ is its Newton height, and that the (LA) condition is satisfied at v when there exists a linear coordinate change which is adapted to ϕ_v at the origin.

If ϕ satisfies (LA) at v, then $h_{\mathrm{lin}}(\phi,v) = h(\phi,v)$, and according to Proposition 3.1.1 the only necessary condition is

$$\frac{1}{p_1'} + \frac{h_{\mathrm{lin}}(\phi,v)}{p_3'} \leqslant \frac{1}{2}. \tag{4.4.3}$$

If ϕ does not satisfy (LA) at v, then from Proposition 1.4.4 we know that this is only possible for the normal form

$$\phi_{v,y}(y) := (y_2 - y_1^2\psi(y_1))^k r(y),$$

where $r(0) \neq 0$, $\psi(0) \neq 0$, and $2 \leqslant k < \infty$, since this is the only non-adapted normal form. Again, Proposition 3.1.1 implies that in this case we have two necessary conditions, namely

$$\frac{1}{p_1'} + \frac{h_{\mathrm{lin}}(\phi, v)}{p_3'} \leqslant \frac{1}{2} \qquad \text{and} \qquad \frac{h(\phi, v)}{p_3'} \leqslant \frac{1}{2}, \tag{4.4.4}$$

where $h(\phi, v) = k$ and $h_{\mathrm{lin}}(\phi, v) = 2k/3$. Note that in the case $h_{\mathrm{lin}}(\phi, v) = h(\phi, v)$ the second condition in (4.4.4) would be redundant. Thus, if we now vary v over the points where $\eta(v) \neq 0$, then we obtain the conditions

$$\frac{1}{p_1'} + \frac{h_{\mathrm{lin}}^{\mathrm{gl}}(\phi)}{p_3'} \leqslant \frac{1}{2} \qquad \text{and} \qquad \frac{h^{\mathrm{gl}}(\phi)}{p_3'} \leqslant \frac{1}{2},$$

where we remind that $h_{\mathrm{lin}}^{\mathrm{gl}}(\phi)$ and $h^{\mathrm{gl}}(\phi)$ are respectively global linear height and global Newton height defined as in (1.4.5).

At all points v where (LA) is satisfied and where $|x|_\kappa^{\mathcal{D}_\mathcal{W}}$ is smooth (i.e., v is not on an axis) we get the local Fourier restriction estimate in the range (4.4.3) directly from Proposition 3.2.2. We shall briefly touch upon what happens in the case when v is situated on the axis in Subsection 4.4.1. In this case one has to only slightly adjust the proofs in Section 4.3.

In the case when (LA) is not satisfied at v let us call the pair $(\mathfrak{p}_1, \mathfrak{p}_3) = (\mathfrak{p}_1(v), \mathfrak{p}_3(v))$ given by

$$\Big(\frac{1}{\mathfrak{p}_1'}, \frac{1}{\mathfrak{p}_3'}\Big) = \Big(\frac{1}{2} - \frac{h_{\mathrm{lin}}(\phi, v)}{2h(\phi, v)}, \frac{1}{2h(\phi, v)}\Big)$$

the critical exponent of ϕ at v. It is obtained as the intersection of the lines

$$\frac{1}{p_1'} + \frac{h_{\mathrm{lin}}(\phi, v)}{p_3'} = \frac{1}{2} \qquad \text{and} \qquad \frac{1}{p_3'} = \frac{1}{2h(\phi, v)}$$

in the $(1/p_1', 1/p_3')$ plane. Thus, for the local estimate in this case it suffices to prove the inequality

$$\|\widehat{f}\|_{L^2(\mathrm{d}\mu_{0,v})} \leqslant C\|f\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})},$$

where

$$\langle \mu_{0,v}, f\rangle = \int f(x_1, x_2, \phi_v(x_1, x_2))\, \eta_v(x_1, x_2)\, \mathrm{d}x_1 \mathrm{d}x_2$$

and

$$\Big(\frac{1}{p_1'}, \frac{1}{p_3'}\Big) \in \Big\{\Big(0, \frac{1}{2h(\phi, v)}\Big), \Big(\frac{1}{2}, 0\Big), \Big(\frac{1}{\mathfrak{p}_1'(v)}, \frac{1}{\mathfrak{p}_3'(v)}\Big)\Big\},$$

since then we get the full range from the necessary conditions by interpolation. We shall only give a sketch of the proof in this case too in Subsections 4.4.2 and 4.4.3, since it is almost identical to a type of singularity considered in Subsection 3.3.5.

4.4.1 Restriction for the adapted case

As mentioned, in the adapted case one needs to prove the Fourier restriction estimate for $(p_1, p_3) \in (1,2)^2$ satisfying

$$\frac{1}{p_1'} + \frac{h_{\mathrm{lin}}(\phi, v)}{p_3'} = \frac{1}{2},$$

and the part of the measure where the amplitude in (4.4.1) is smooth the restriction estimate is already proven in Chapter 3.

Now the amplitude in (4.4.1) (in particular the function $x \mapsto |x|_\kappa^{\mathcal{D}_\mathcal{W}}$) is in general not smooth along the axes $x_1 = 0$ and $x_2 = 0$. Namely, on the $x_1 = 0$ axis one can take only the derivatives (of the amplitude) in the x_2 direction, and analogously on the $x_2 = 0$ axis one can take only derivatives in the x_1 direction. Note that the only possible non-adapted normal form appears only away from the axes.

Let us consider without loss of generality what happens for the point $v = (v_1, 0)$ on the axis $x_2 = 0$ and its associated measure $\mu_{0,v}$ defined in (4.4.2). We shall only briefly sketch what one needs to do in order to prove the Fourier restriction estimate when the amplitude is not smooth in the x_2 direction at v. Since we are dealing only with adapted normal forms, it suffices to obtain an appropriate estimate on the Fourier transform, after which one can apply Lemma 4.3.2 or its modification such as Lemma 2.3.1. Often we shall need to use the Littlewood-Paley theorem in order to localize away from the axis.

According to the normal forms listed at the end of Subsection 4.2.1, and under the condition (H1) (see Section 1.4 for the formulation of this condition), we have the following cases.

Case 1. If (under the notation of Section 4.2) we have $k = \infty$, then by the considerations from Subsection 4.2.2 the phase at v is

$$\phi_v(x - v) = (x_1 - v_1)^{\tilde{k}} + \varphi(x_1, x_2),$$

where $2 \leqslant \tilde{k} < \infty$ and φ is a flat function at v. This corresponds to Normal form (i.y2) and we have $h_{\mathrm{lin}}(\phi, v) = \tilde{k}$. Since $|x|_\kappa^{\mathcal{D}_\mathcal{W}}$ is still smooth in the x_1 direction, one can use the van der Corput lemma in the x_1 direction and get that the decay of the Fourier transform of $\mu_{0,v}$ is $(1 + |\xi|)^{-1/\tilde{k}}$. This now implies the desired estimate (see the result [56, Theorem 1.2] or the results in Section 2.3, or apply an appropriate modification of Lemma 4.3.2).

If $2 \leqslant k < \infty$, then we have three further cases.

Case 2. Let us consider the phase

$$\phi_v(x) = x_2^k r(x),$$

where $r(v) \neq 0$ and $k \geqslant 2$. In this case the linear height is $h_{\mathrm{lin}}(\phi, v) = k$. Here the idea is to apply the Littlewood-Paley theorem in order to localize away from the axis $x_2 = 0$, and rescale afterwards. Since the essentially same thing was done in Section 4.3 for this type of singularity (see the proof for Normal form (i) in Subsection 4.3.1), let us just briefly mention the main differences compared to there. Obviously, one scales differently the measure pieces away from the axis obtained by applying the Littlewood-Paley theorem since here we consider different exponents (p_1, p_3). The main difference is that we do not use the Hessian determinant to obtain a decay on the Fourier transformation of the rescaled measure piece (since the Hessian determinant may vanish of infinite order as only (H1) is assumed and not the stronger condition (H2)), but rather directly from the form of the phase above. This we may now do since the new amplitude for the rescaled measure pieces is now smooth.

Case 3. Let us now consider the case when the phase is nondegenerate, i.e., the Hessian determinant does not vanish at v (and in particular $h_{\mathrm{lin}}(\phi, v) = 1$). Here we use the Littlewood-Paley theorem as in Case 2, but after rescaling we use the size of the Hessian determinant of the new phase to get a decay on the Fourier transform of the measure (as was done in Section 4.3 for Normal forms (i), (ii), and (iii)).

Case 4. The final case is when (after an affine change to y or w coordinates from Section 4.2) we have

$$\phi_{v,u}(u) = u_1^2 r_1(u) + u_2^{k_0} r_2(u),$$

where $3 \leqslant k_0 \leqslant \infty$, $r_1(0) \neq 0$, and in case when $k_0 < \infty$ then $r_2(0) \neq 0$ and $h_{\mathrm{lin}}(\phi, v) = 2k_0/(2 + k_0)$. If $k_0 = \infty$ then $h_{\mathrm{lin}}(\phi, v) = 2$, and the above equality holds in the sense that we can take any $k_0 \geqslant 0$ and r_2 flat. Inspecting the y and w coordinates from Section 4.2 we see that the $x_2 = 0$ axis corresponds to the $u_2 = 0$ axis.

If $k_0 = \infty$, we can argue in the same way as in the case $k = \infty$ above (here it is critical that $\partial_{u_1} = c\partial_{x_1}, c \neq 0$, in order to be able to apply the van der Corput lemma in the smooth direction).

Otherwise, if k_0 is finite, we proceed again with a Littlewood-Paley decomposition in the u_2 direction (as was done in Subsection 4.3.3 for Normal forms (ii) and (iii)) in order to get a smooth amplitude. At this point one gets that the estimate on the decay of the Fourier transform is $2^{k_0 l/2}(1 + |\xi|)^{-1}$ by using the size of the Hessian determinant. Since the new rescaled phase is (compare with (4.3.7))

$$u_1^2 r_1(u_1, 2^{-l}u_2) + 2^{-k_0 l} u_2^{k_0} r_2(u_1, 2^{-l}u_2),$$

by applying the van der Corput lemma in u_1 we also have the decay estimate $(1+|\xi|)^{-1/2}$. Interpolating these two estimates gives the decay $2^l(1 + |\xi|)^{-(2+k_0)/(2k_0)}$, which turns out to be precisely what one needs when interpolating with the Plancherel estimate.

4.4.2 Restriction for the non-adapted case: preliminaries

Let us fix a phase function ϕ_{loc} of the form

$$\phi_{\mathrm{loc}}(x) = (x_2 - x_1^2\psi(x_1))^k r(x),$$

where $\psi(0), r(0) \neq 0$ and $k \in \mathbb{N}$, $k \geqslant 2$. The adapted coordinates are obtained by the smooth transformation $y_1 = x_1$, $y_2 = x_2 - x_1^2\psi(x_1)$:

$$\phi^a_{\mathrm{loc}}(y) \coloneqq y_2^k r^a(y),$$

where $r^a(0) \neq 0$. Thus, the Newton height of ϕ_{loc} is k and the Newton distance is $d \coloneqq 2k/3$ (which coincides with the linear height h_{lin}). The Varchenko exponent is 0 since in adapted coordinates the principal face is noncompact. Then from e.g. Section 2.3 we know that we automatically have the Fourier restriction estimate

$$\| \mathcal{F} f\|_{L^2(\mathrm{d}\nu)} \lesssim \|f\|_{L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})}, \qquad f \in \mathcal{S}(\mathbb{R}^3), \tag{4.4.5}$$

for the exponents

$$\Big(\frac{1}{p_1'}, \frac{1}{p_3'}\Big) = \Big(0, \frac{1}{2k}\Big) \qquad \text{and} \qquad \Big(\frac{1}{p_1'}, \frac{1}{p_3'}\Big) = \Big(\frac{1}{2}, 0\Big),$$

and where the measure ν is defined through

$$\langle \nu, f\rangle = \int f(x_1, x_2, \phi_{\mathrm{loc}}(x_1, x_2))\, a(x_1, x_2)\, \mathrm{d}x_1 \mathrm{d}x_2, \tag{4.4.6}$$

where $a \in C_c^\infty(\mathbb{R}^2)$ is a nonnegative function supported in a small neighbourhood of the origin. It remains to obtain the Fourier restriction estimate for the critical exponent, which in this case is

$$\Big(\frac{1}{\mathfrak{p}_1'}, \frac{1}{\mathfrak{p}_3'}\Big) = \Big(\frac{1}{6}, \frac{1}{2k}\Big). \tag{4.4.7}$$

The case $k = 2$ has been solved in Chapter 3. In the case $k = 3$ the critical exponent lies on the diagonal and so this case has already been solved in [51].

In the case $k \geqslant 4$ we have $1/\mathfrak{p}_1' > 1/\mathfrak{p}_3'$ and so one would need to slightly modify the methods used in Section 3.3 (i.e., the methods for the case $h_{\mathrm{lin}}(\phi) < 2$) since there one interpolated between two points of the form

$$\Big(\frac{1}{p_1'}, \frac{1}{p_3'}\Big) = \Big(0, \frac{s}{2}\Big) \qquad \text{and} \qquad \Big(\frac{1}{p_1'}, \frac{1}{p_3'}\Big) = \Big(\frac{1}{2}, \frac{1}{2}\Big),$$

for some $0 < s < 1/k$. In the case $1/\mathfrak{p}_1' > 1/\mathfrak{p}_3'$ in general one would need to interpolate between three points

$$\Big(\frac{1}{p_1'}, \frac{1}{p_3'}\Big) = \Big(0, 0\Big), \qquad \Big(\frac{1}{p_1'}, \frac{1}{p_3'}\Big) = \Big(\frac{1}{2}, \frac{1}{2}\Big), \qquad \text{and} \qquad \Big(\frac{1}{p_1'}, \frac{1}{p_3'}\Big) = \Big(\frac{1}{2}, 0\Big).$$

In particular, if one has an operator $T : L^p \to L^{p'}$, where here we denote $L^p = L^{p_3}_{x_3}(L^{p_1}_{(x_1,x_2)})$ for $p = (p_1, p_3)$, satisfying the estimates

$$\begin{aligned} \|T\|_{L^p \to L^{p'}} &\lesssim A_1 \qquad \text{for } \Big(\frac{1}{p_1'}, \frac{1}{p_3'}\Big) = \Big(0, 0\Big), \\ \|T\|_{L^p \to L^{p'}} &\lesssim A_2 \qquad \text{for } \Big(\frac{1}{p_1'}, \frac{1}{p_3'}\Big) = \Big(\frac{1}{2}, \frac{1}{2}\Big), \\ \|T\|_{L^p \to L^{p'}} &\lesssim A_3 \qquad \text{for } \Big(\frac{1}{p_1'}, \frac{1}{p_3'}\Big) = \Big(\frac{1}{2}, 0\Big), \end{aligned} \tag{4.4.8}$$

then by interpolation one has the estimate

$$\|T\|_{L^p \to L^{p'}} \lesssim A_1^{2/3}\, A_2^{1/k}\, A_3^{(k-3)/(3k)} \qquad \text{for } \Big(\frac{1}{p_1'}, \frac{1}{p_3'}\Big) = \Big(\frac{1}{6}, \frac{1}{2k}\Big).$$

In our special case we shall not use the above general approach since we recall that when we considered the case when the mitigating factor was present (to be more precise, the case of Normal form (vi) considered in Subsection 4.3.6), after performing some decompositions and rescalings one got measure pieces for which one needed to prove the Fourier restriction estimate for the exponent

$$\Big(\frac{1}{p_1'}, \frac{1}{p_3'}\Big) = \Big(\frac{1}{6}, \frac{1}{3}\Big). \tag{4.4.9}$$

In the current case without the mitigating factor it turns out that we shall get the same measure pieces, but for which we need to prove the Fourier restriction estimate for the exponent (4.4.7). Thus, if we have the Fourier restriction estimate for the exponent (4.4.9), then the Fourier restriction for (4.4.7) is obtained by interpolating with the result for

$$\Big(\frac{1}{p_1'}, \frac{1}{p_3'}\Big) = \Big(\frac{1}{6}, 0\Big),$$

which one can obtain by applying the 2-dimensional Fourier restriction result for curves with nonvanishing curvature.

These stronger estimates for the rescaled measure pieces do not contradict the necessary conditions obtained by Knapp-type examples in Section 3.1 since the information on the exponents and the Newton height of ϕ is consumed in the rescaling procedure (which is different in this section and in Subsection 4.3.6).

Let us begin with some preliminary reductions. By the results from Section 3.2.2, instead of considering the whole measure (4.4.6), we may reduce ourselves to considering the part near the principal root jet in the half plane $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geqslant 0\}$:

$$\langle \nu^{\rho_1}, f \rangle = \int_{x_1 \geqslant 0} f(x, \phi_{\mathrm{loc}}(x))\, a(x)\, \rho_1(x) \mathrm{d}x,$$

where

$$\rho_1(x) = \chi_0\Big(\frac{x_2 - \psi(0)x_1^2}{\varepsilon x_1^2}\Big)$$

for an ε which we can take to be as small as we want.

The next step is to use a Littlewood-Paley argument in the (x_1, x_2)-plane and the scaling by $\tilde\kappa$ dilations

$$\delta_r^{\tilde\kappa}(x) = (r^{\tilde\kappa_1} x_1, r^{\tilde\kappa_2} x_2)$$

where $\tilde\kappa := (1/(2k), 1/k)$ is the weight associated to the principal face of ϕ_{loc}. Then one is reduced to proving (4.4.5) for the measures

$$\langle \nu_j, f \rangle = \int f(x, \phi(x, \delta))\, a(x, \delta)\, \mathrm{d}x,$$

uniformly in j, where the function $\phi(x, \delta)$ has the form

$$\phi(x, \delta) := (x_2 - x_1^2 \psi(\delta_1 x_1))^k r(\delta_1 x_1, \delta_2 x_2),$$

where

$$\delta = (\delta_1, \delta_2) := (2^{-\tilde{\kappa}_1 j}, 2^{-\tilde{\kappa}_2 j}).$$

Note that we can take $|\delta| \ll 1$. The amplitude $a(x, \delta) \geqslant 0$ is a smooth function of (x, δ) supported where

$$x_1 \sim 1 \sim |x_2|.$$

We may additionally assume $|x_2 - x_1^2 \psi(0)| \ll 1$ due to ρ_1, and by compactness we may in fact reduce ourselves to assuming $|(x_1, x_2) - (v_1^0, v_2^0)| \ll 1$ for some $(v_1^0, v_2^0) \in \mathbb{R}^2$ with $v_1^0 \sim 1$.

The following step is to again apply the Littlewood-Paley theorem, but this time in the x_3 direction (for which the mixed norm Littlewood-Paley theory is needed from e.g. [62]), and reduce the Fourier restriction problem for ν_j to the Fourier restriction for the measures

$$\langle \nu_{\delta,l}, f \rangle = \int f(x, \phi(x, \delta)) \, \chi_1(2^{kl} \phi(x, \delta)) \, a(x, \delta) \, \mathrm{d}x,$$

i.e., we need to prove

$$\| \mathcal{F} f \|_{L^2(\mathrm{d}\nu_{\delta,l})} \lesssim \| f \|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})}, \qquad f \in \mathcal{S}(\mathbb{R}^3),$$

uniformly in l and δ, where $l \gg 1$ and $|\delta| \ll 1$.

Finally, we perform a change of coordinates and a rescaling. Namely, after substituting $(x_1, x_2) \mapsto (x_1, 2^{-l} x_2 + x_1^2 \psi(\delta_1 x_1))$ we get

$$\langle \nu_{\delta,l}, f \rangle = 2^{-l} \int f(x_1, 2^{-l} x_2 + x_1^2 \psi(\delta_1 x_1), 2^{-kl} \phi^a(x, \delta, l)) \, a(x, \delta, l) \, \mathrm{d}x,$$

where

$$\begin{aligned} a(x, \delta, l) &:= \chi_1(\phi^a(x, \delta, l)) \, a(x_1, 2^{-l} x_2 + x_1^2 \psi(\delta_1 x_1), \delta), \\ \phi^a(x, \delta, l) &:= x_2^k r(\delta_1 x_1, \delta_2(2^{-l} x_2 + x_1^2 \psi(\delta_1 x_1))). \end{aligned}$$

Note that $a(x, \delta, l)$ is again supported in a domain where $x_1 \sim 1 \sim |x_2|$. Rescaling we obtain that the Fourier restriction estimate for $\nu_{\delta,l}$ is equivalent to the estimate

$$\| \mathcal{F} f \|_{L^2(\mathrm{d}\tilde{\nu}_{\delta,l})} \lesssim \| f \|_{L^{\mathfrak{p}_3}_{x_3}(L^{\mathfrak{p}_1}_{(x_1,x_2)})}, \qquad f \in \mathcal{S}(\mathbb{R}^3),$$

for the measure

$$\langle \tilde{\nu}_{\delta,l}, f \rangle = \int f(x_1, 2^{-l} x_2 + x_1^2 \psi(\delta_1 x_1), \phi^a(x, \delta, l)) \, a(x, \delta, l) \, \mathrm{d}x. \tag{4.4.10}$$

As mentioned, since this measure is of the same form as (4.3.21), we are interested in proving the stronger estimate

$$\| \mathcal{F} f \|_{L^2(\mathrm{d}\tilde{\nu}_{\delta,l})} \lesssim \| f \|_{L^{\tilde{\mathfrak{p}}_3}_{x_3}(L^{\tilde{\mathfrak{p}}_1}_{(x_1,x_2)})}, \qquad f \in \mathcal{S}(\mathbb{R}^3),$$

where

$$\Big(\frac{1}{\tilde{\mathfrak{p}}_1'},\frac{1}{\tilde{\mathfrak{p}}_3'}\Big) := \Big(\frac{1}{6},\frac{1}{3}\Big).$$

Note that we automatically have the estimate for

$$\Big(\frac{1}{p_1'},\frac{1}{p_3'}\Big) = \Big(\frac{1}{6},0\Big)$$

by a classical result of Zygmund [89], since $x_1 \mapsto (x_1, 2^{-l}x_2 + x_1^2\psi(\delta_1 x_1))$ is a curve with curvature bounded from below uniformly in $|x_2| \sim 1$, $2^{-l} \ll 1$, and $\delta_1 \ll 1$.

4.4.3 Restriction for the non-adapted case: spectral decomposition

We begin by performing a spectral decomposition of the measure $\tilde{\nu}_{\delta,l}$. For $(\lambda_1,\lambda_2,\lambda_3)$ dyadic numbers with $\lambda_i \geqslant 1$, $i = 1,2,3$, we consider localized measures ν_l^λ defined through

$$\begin{aligned}\widehat{\nu_l^\lambda}(\xi) =&\chi_1\Big(\frac{\xi_1}{\lambda_1}\Big)\,\chi_1\Big(\frac{\xi_2}{\lambda_2}\Big)\,\chi_1\Big(\frac{\xi_3}{\lambda_3}\Big)\\ &\times\int e^{-i\Phi(x,\delta,l,\xi)}\,a(x,\delta,l)\,\chi_1(x_1)\,\chi_1(x_2)\,\mathrm{d}x,\end{aligned} \tag{4.4.11}$$

where the phase function is

$$\Phi(x,\delta,l,\xi) := \xi_3\phi^a(x,\delta,l) + 2^{-l}\xi_2 x_2 + \xi_2 x_1^2\psi(\delta_1 x_1) + \xi_1 x_1. \tag{4.4.12}$$

By an abuse of notation, above whenever $\lambda_i = 1$, we consider the cutoff function $\chi_1(\xi_i/\lambda_i)$ to be actually $\chi_0(\xi_1/\lambda_1)$, i.e., it localizes so that $|\xi_i| \lesssim 1$.

Let us introduce the convolution operators $\tilde{T}_{\delta,l}f := f * \widehat{\tilde{\nu}}_{\delta,l}$ and $T_l^\lambda f := f * \widehat{\nu_l^\lambda}$. Then we need to show

$$\|\tilde{T}_{\delta,l}\|_{L^{\tilde{\mathfrak{p}}}\to L^{\tilde{\mathfrak{p}}'}} \lesssim 1,$$

since $\tilde{T}_{\delta,l}$ is the "R^*R" operator, i.e., one has $\tilde{T}_{\delta_l} = (\tilde{R}_{\delta,l})^*\tilde{R}_{\delta,l}$ if $\tilde{R}_{\delta,l}$ denotes the Fourier restriction operator with respect to the surface carried measure $\tilde{\nu}_{\delta,l}$. Therefore, the boundedness of $\tilde{T}_{\delta,l}$ is equivalent to the boundedness of $\tilde{R}_{\delta,l}$ by Hölder's inequality.

Our first step shall be to reduce the problem to the case when $\lambda_2 \ll 2^l$. In order to achieve this we split the Fourier transform of $\tilde{\nu}_{\delta,l}$ as

$$\widehat{\tilde{\nu}}_{\delta,l} = (1-\chi_0(2^{-l}\xi_2))\widehat{\tilde{\nu}}_{\delta,l} + \chi_0(2^{-l}\xi_2)\widehat{\tilde{\nu}}_{\delta,l}, \tag{4.4.13}$$

where we assume that χ_0 is supported in a sufficiently small neighbourhood of the origin, and we denote the respective operators for the respective terms by T_I and T_{II}.

For the first term in (4.4.13) and its operator T_I one uses Lemma 4.3.2 above, though with a slight modification. First, since on the support of $(1-\chi_0(2^{-l}\xi_2))\widehat{\tilde{\nu}}_{\delta,l}$ we have $|\xi_2| \gtrsim 2^l$, one can easily show by using (4.4.12) that now

$$|(1-\chi_0(2^{-l}\xi_2))\widehat{\tilde{\nu}}_{\delta,l}| \lesssim 2^{-l/2}(1+|\xi_3|)^{-1},$$

as the "worst case" is when $|\xi_1| \sim |\xi_2|$ and $|\xi_3| \sim |2^{-l}\xi_2|$, in which case we use stationary phase in both x_1 and x_2 (and in other cases we get a better decay by integrating by parts). In order to obtain the Plancherel estimate $L^1(\mathbb{R}; L^2(\mathbb{R}^2)) \to L^\infty(\mathbb{R}; L^2(\mathbb{R}^2))$ in Lemma 4.3.2 for T_I it suffices to prove it for T_{II} and $\tilde{T}_{\delta,l}$ (formally, one needs to actually consider the $L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ estimate for a fixed ξ_3). For the operator $\tilde{T}_{\delta,l}$ we get the bound 2^l in the same way as in Lemma 4.3.2. The main fact to notice is that in (4.4.10) the Jacobian of $(x_1, x_2) \mapsto (x_1, 2^{-l}x_2 + x_1^2\psi(\delta_1 x_1))$ is of size 2^{-l}. One now gets the same estimate automatically for T_{II} since the L^1 norm of the Fourier transform of the cutoff function $\chi_0(2^{-l}\xi_2)$ is of size ~ 1. The estimate $L^{\tilde{\mathfrak{p}}} \to L^{\tilde{\mathfrak{p}}'}$ estimate for T_I follows with constant of size $\sim 1 = (2^{-l/2})^{2/3}(2^l)^{1/3}$.

For the operator T_{II} we shall use the spectral decomposition (4.4.11) where we may now assume $\lambda_2 \ll 2^l$. Recall that for an operator of the form $Tf = f * \widehat{g}$ the A_1 constant from (4.4.8) is bounded by the L^∞ norm of $\widehat{g}$, and the A_2 constant is bounded by the L^∞ norm of g. If we now furthermore have that $\widehat{g}$ has its support in the ξ_3 coordinate localized at $|\xi_3| \lesssim \lambda_3$, then by Lemma 2.3.2 we have the estimate

$$\|T\|_{L^p \to L^{p'}} \lesssim A_1 \lambda_3^{1/2}, \qquad \text{for } \Big(\frac{1}{p_1'}, \frac{1}{p_3'}\Big) = \Big(0, \frac{1}{4}\Big),$$

and so by interpolation we get

$$\|T\|_{L^{\tilde{\mathfrak{p}}} \to L^{\tilde{\mathfrak{p}}'}} \lesssim A_1^{2/3} A_2^{1/3} \lambda_3^{1/3}. \tag{4.4.14}$$

The inverse Fourier transform of (4.4.11) is

$$\begin{aligned} \nu_l^\lambda(x) = \lambda_1\lambda_2\lambda_3 \int & \check{\chi}_1(\lambda_1(x_1 - y_1))\, \check{\chi}_1(\lambda_2(x_2 - 2^{-l}y_2 - y_1^2\psi(\delta_1 y_1))) \\ & \times \check{\chi}_1(\lambda_3(x_3 - \phi^a(y, \delta, l)))\, a(y, \delta, l)\, \chi_1(y_1)\, \chi_1(y_2)\, \mathrm{d}y. \end{aligned} \tag{4.4.15}$$

One can consider either the substitution $(z_1, z_2) = (\lambda_1 y_1, \lambda_2 2^{-l} y_2)$, or the substitution $(z_1, z_2) = (\lambda_1 y_1, \lambda_3 \phi^a(y, \delta, l))$ (in order to carry this out one needs to consider the cases $y_2 \sim 1$ and $y_2 \sim -1$ separately), and get

$$\|\nu_j^\lambda\|_{L^\infty} \lesssim \min\{2^l \lambda_3, \lambda_2\}.$$

But now since $\lambda_2 \ll 2^l$ we may take $A_2 := \lambda_2$.

It remains to calculate the L^∞ bound for the $\widehat{\nu_l^\lambda}$ function. This we can do by estimating the oscillatory integral in (4.4.11). As the calculations for the oscillatory integral in this case are almost identical to the ones in Subsection 3.3.5, we shall only briefly explain the case when $\lambda_1 \sim \lambda_2$, $2^{-l}\lambda_2 \ll \lambda_3 \ll \lambda_2$, corresponding to Case 6 in Subsection 3.3.5. In all the other cases one gets that one can sum absolutely in the operator norm the operator pieces T_l^λ.

Let us remark that since $\lambda_2 \ll 2^l$, the case when $\lambda_1 \sim \lambda_2$, $2^{-l}\lambda_2 \sim \lambda_3$, corresponding to Case 4 in Subsection 3.3.5, does not appear anymore. This is critical since in this case one would not have absolute summability, nor would the complex interpolation method developed in [51] work. This is the reason why we needed to consider T_I and T_{II} separately.

Case $\lambda_1 \sim \lambda_2$ and $2^{-l}\lambda_2 \ll \lambda_3 \ll \lambda_2$. As was obtained in Subsection 3.3.5, we have

$$\|\widehat{\nu_l^\lambda}\|_{L^\infty} \lesssim \lambda_1^{-1/2}\lambda_3^{-N} \tag{4.4.16}$$

for any $N > 0$, that is, we have $A_1 = \lambda_1^{-1/2}\lambda_3^{-N}$, and recall that $A_2 = \lambda_2$, Therefore (4.4.14) gives

$$\|T_l^\lambda\|_{L^{\tilde{\mathfrak{p}}} \to L^{\tilde{\mathfrak{p}}'}} \lesssim \lambda_3^{-N}.$$

In order to be able to sum in $\lambda_1 \sim \lambda_2$ we need to use the complex interpolation method from [51]. For a fixed λ_3 and ζ a complex number we define the measure $\mu_\zeta^{\lambda_3}$ by

$$\mu_\zeta^{\lambda_3} := \gamma(\zeta) \sum_{\lambda_1,\lambda_2} \lambda_1^{(1-3\zeta)/2}\nu_l^\lambda,$$

where the sum is over $\lambda_3 \ll \lambda_2 \ll 2^l$ and $\lambda_1 \sim \lambda_2$, and where $\gamma(\zeta) = 2^{-3(\zeta-1)/2} - 1$. We denote the associated convolution operator by $T_\zeta^{\lambda_3}$ and we recover with $\zeta = 1/3$ the operator we want to estimate.

By a complex interpolation argument it suffices to show that

$$\begin{aligned}
\|T_{it}^{\lambda_3}\|_{L^p \to L^{p'}} &\lesssim \lambda_3^{-N}, && \text{for } \Big(\frac{1}{p_1'}, \frac{1}{p_3'}\Big) = \Big(0, \frac{1}{4}\Big),\\
\|T_{1+it}^{\lambda_3}\|_{L^p \to L^{p'}} &\lesssim 1, && \text{for } \Big(\frac{1}{p_1'}, \frac{1}{p_3'}\Big) = \Big(\frac{1}{2}, \frac{1}{2}\Big).
\end{aligned}$$

for some $N > 0$, with constants uniform in $t \in \mathbb{R}$. The first estimate follows directly from the fact that $\widehat{\nu_l^\lambda}$ have essentially disjoint supports with respect to λ and the estimate (4.4.16) (see Lemma 2.3.1, (i)), and for the other bound we need to estimate the L^∞ norm of the corresponding sum of the expressions (4.4.15). The proof is the same as in Subsection 3.3.5, Case 6, up to the formal difference in the function ϕ^a which here behaves like y_2^k, and there like y_2^2. Since the domain of integration in (4.4.15) is $|y_2| \sim 1$, this is not essential. This finishes (the sketch of) the proof of the Fourier restriction for the non-adapted case, and also the proof of Theorem 1.4.2.

4.5 Application to PDEs and ΨDEs

Recall that we consider the pseudodifferential equation

$$\begin{cases} (\partial_t - i\phi(D))u(x,t) &= F(x,t), \quad (x,t) \in \mathbb{R}^2 \times (0,\infty),\\ \qquad\qquad u(x,0) &= G(x), \qquad x \in \mathbb{R}^2, \end{cases}$$

for $F \in \mathcal{S}(\mathbb{R}^3)$, $G \in \mathcal{S}(\mathbb{R}^2)$, where ϕ, $\mathcal{W}$, and $(p_1, p_3) \in (1,2)^2$ are either as in Theorem 1.4.1 or Theorem 1.4.2, and where we additionally assume $\mathcal{D} \in \{0,1\}$. Note that ϕ is locally bounded and has polynomial growth at infinity, and note that according to Remark 4.1.3

the weight $\mathcal{W}$ is locally integrable in $\mathbb{R}^2$. The formula for a solution of the above equation is obtained through the Duhamel principle:

$$u(x,t) = (e^{i\phi(D)t}G)(x) + \int_0^t (e^{i\phi(D)(t-s)}F(\cdot,s))(x,s)\mathrm{d}s. \tag{4.5.1}$$

Note that $u \in C^\infty(\mathbb{R}^2 \times \mathbb{R}) \cap L^\infty_t((C_0)_{(x_1,x_2)}(\mathbb{R}^2))$, where C_0 denotes the space of continuous functions which tend to 0 at infinity.

We consider the following two surface carried measures (the second defined as in (4.0.3)):

$$\begin{aligned}\langle \mu_\phi, f\rangle &= \int_{\mathbb{R}^2\setminus\{0\}} f(x_1,x_2,\phi(x_1,x_2))\,\mathrm{d}x,\\ \langle \mu, f\rangle &= \int_{\mathbb{R}^2\setminus\{0\}} f(x_1,x_2,\phi(x_1,x_2))\,\mathcal{W}(x_1,x_2)\,\mathrm{d}x,\end{aligned}$$

and we assume that the Fourier restriction estimate (4.0.2) for μ holds true for $(p_1,p_3) \in (1,2)^2$. One can easily check that

$$(e^{i\phi(D)t}G)(x) = \mathcal{F}^{-1}((\mathcal{F}\,G)\mathrm{d}\mu_\phi)(x,t) = \mathcal{F}^{-1}(\mathcal{W}^{-1}(\mathcal{F}\,G)\mathrm{d}\mu)(x,t),$$

and so this is precisely the Fourier extension operator of μ applied to the function $\mathcal{W}^{-1}\,\mathcal{F}\,G$. We can therefore bound the $L^{p_3'}_t(L^{p_1'}_{(x_1,x_2)})$ norm of this expression by the $L^2(\mathrm{d}\mu)$ norm of $\mathcal{W}^{-1}\,\mathcal{F}\,G$.

It remains to estimate the $L^{p_3'}_t(L^{p_1'}_{(x_1,x_2)})$ norm of the second term in (4.5.1). It turns out that the operator associated to this second term is closely related to the operator $f \mapsto f * \mathcal{F}^{-1}\,\mu$ (which we know is bounded from $L^{p_3}_t(L^{p_1}_{(x_1,x_2)})$ to $L^{p_3'}_t(L^{p_1'}_{(x_1,x_2)})$ since this is the corresponding R^*R operator). Namely, one can check that

$$\int_0^\infty (e^{i\phi(D)(t-s)}F(\cdot,s))(x,s)\mathrm{d}s = \Big((F\,\chi_{(0,\infty)}(s)) * (\mathcal{F}^{-1}\,\mu_\phi)\Big)(x,t),$$

and therefore it remains to pass from μ_ϕ to μ and to pass from integrating over $(0,\infty)$ in s to integrating over $(0,t)$ in s.

In order to do this, our first step is to use the Littlewood-Paley theorem in the x-direction so that our problem is reduced to proving the boundedness of the operator

$$\int_0^t (e^{i\phi(D)(t-s)}\,\eta_j(D)\,F(\cdot,s))(x,s)\mathrm{d}s \tag{4.5.2}$$

where $(\eta_j)_{j\in\mathbb{Z}}$, $\eta_j = \eta \circ \delta_{2^{-j}}$, constitutes a partition of unity in $\mathbb{R}^2\setminus\{0\}$ (as in (4.1.1) in Subsection 4.1.1) respecting the κ-mixed homogeneous dilation $\delta_{2^{-j}}$ defined in (4.0.1). By unwinding the definition of the operator in (4.5.2) and inserting the $\mathcal{W}$ factor, one obtains the expression (up to a universal constant)

$$\int_0^t \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} e^{i(x-y)\cdot\xi + i(t-s)\phi(\xi)} \eta_j(\xi)\mathcal{W}(\xi)\mathrm{d}\xi \right) F_{\mathcal{W}^{-1}}(y,s)\mathrm{d}y\mathrm{d}s, \tag{4.5.3}$$

where $F_{\mathcal{W}^{-1}} = \mathcal{F}^{-1}{}_{(x_1,x_2)}(\mathcal{W}^{-1}\,\mathcal{F}_{(x_1,x_2)}\,F)$. The expression within the brackets defines a convolution kernel $K_j(t-s;x-y)$ whose associated operator $T_j(t-s)$ in the x variable is a bounded mapping from $L^q(\mathbb{R}^2)$ to $L^{q'}(\mathbb{R}^2)$ for any $q \in [1,2]$ (since the integrand in the brackets is an $L^\infty_c(\mathbb{R}^2)$ function). Using the dominated convergence theorem one can get strong continuity of the operator valued function $T_j : \mathbb{R} \to \mathcal{L}(L^q(\mathbb{R}^2); L^{q'}(\mathbb{R}^2))$ (which in turn, by the uniform boundedness principle, implies joint continuity $T_j : \mathbb{R} \times L^q(\mathbb{R}^2) \to L^{q'}(\mathbb{R}^2)$).

We may now apply the Christ-Kiselev lemma (for a proof of this variant see e.g. [74, Chapter IV, Lemma 2.1]):

Lemma 4.5.1. *Let Y and Z be separable Banach spaces and let $K : \mathbb{R} \to \mathcal{L}(Y,Z)$ be a continuous function from the real numbers to the space of bounded linear mappings $Y \to Z$ equipped with the strong operator topology. If the operator defined by*

$$(Tf)(t) := \int_{\mathbb{R}} K(t-s)f(s)\mathrm{d}s$$

is a bounded mapping from $L^q(\mathbb{R},Y)$ to $L^{q'}(\mathbb{R},Z)$ for some $q \in (1,2)$, then the operator defined by

$$(Wf)(t) := \int_{-\infty}^{t} K(t-s)f(s)\mathrm{d}s$$

is also a bounded mapping from $L^q(\mathbb{R},Y)$ to $L^{q'}(\mathbb{R},Z)$, and in particular

$$\|W\|_{L^q(\mathbb{R},Y)\to L^{q'}(\mathbb{R},Z)} \lesssim_q \|T\|_{L^q(\mathbb{R},Y)\to L^{q'}(\mathbb{R},Z)}.$$

Then we get that the $L^{p_3}_t(L^{p_1}_{(x_1,x_2)}) \to L^{p_3'}_t(L^{p_1'}_{(x_1,x_2)})$ boundedness of the operator in (4.5.3) (acting on $F_{\mathcal{W}^{-1}}$) is implied by the $L^{p_3}_t(L^{p_1}_{(x_1,x_2)}) \to L^{p_3'}_t(L^{p_1'}_{(x_1,x_2)})$ boundedness of the operator

$$\int_0^\infty \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} e^{i(x-y)\cdot\xi + i(t-s)\phi(\xi)} \eta_j(\xi)\mathcal{W}(\xi)\mathrm{d}\xi \right) F_{\mathcal{W}^{-1}}(y,s)\mathrm{d}y\mathrm{d}s$$
$$= \Big((F_{\mathcal{W}^{-1}}\,\chi_{(0,\infty)}(s)) * (\mathcal{F}^{-1}\,\mu_j) \Big)(x,t),$$

with essentially the same operator constant bound (up to a multiplicative factor which depends only on $p_3 \in (1,2)$). Here μ_j is the localized measure defined in the same way as in (4.1.2), and recall that this convolution operator is bounded (uniformly in j). This finishes the proof of Corollary 1.4.5.

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Index

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- Lj. Palle, *Mixed norm Strichartz-type estimates for hypersurfaces in three dimensions.* preprint arXiv:1905.09529, online erschienen in Math. Zeitschrift, DOI: 10.1007/s00209-020-02568-8.

- Lj. Palle, *Strichartz estimates for mixed homogeneous surfaces in three dimensions.* preprint arXiv:2004.07751.

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